EXTREMAL PROBLEMS IN BERGMAN SPACES

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2011

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ACKNOWLEDGEMENTS

First, I wish to thank Prof. Peter Duren for all of his help and advice, for all the time he devoted to me, and for all that he has taught me. I wish to thank Gay Duren for her kindness. I also want to thank my other committee members for their time and for the many things they taught me. I appreciate the friendships I have with my fellow graduate students, as well as the rides to the airport many of them have given me, and all the good conversations about mathematics I have had with them.

I am grateful to Paige for all of her support these past five years. In addition, I would like to thank my parents Jim and Barb, my grandfather Thurston, and the other members of my family for their support, including Aunt Linda, Aunt Jean, Uncle Nelson, and Nancy. I am also grateful to Prof. Joe Conlon and his wife Barbara for all the interesting conversations and all the help they gave me, and to Sean and Kathleen Conlon for teaching me so much. In addition, I want to acknowledge the support given to me by Fr. Jeff Njus and Fr. Bill Ashbaugh. *AMDG*.

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ABSTRACT

We deal with extremal problems in Bergman spaces. If A^p denotes the Bergman space, then for any given functional $\phi \neq 0$ in the dual space $(A^p)^*$, an extremal function is a function $F \in A^p$ such that $\|F\|_{A^p} = 1$ and $\operatorname{Re} \phi(F)$ is as large as possible.

We give a simplified proof of a theorem of Ryabykh stating that if k is in the Hardy space H^q for $1/p + 1/q = 1$, and the functional ϕ is defined by

$$\phi(f) = \int_{\mathbb{D}} f(z)\overline{k(z)}\, d\sigma, \qquad f \in A^p,$$

where σ is normalized Lebesgue area measure, then the extremal function over the space A^p is actually in H^p.

We also extend Ryabykh's theorem in the case where p is an even integer. Let p be an even integer, and let ϕ be defined as above. Furthermore, let p_1 and q_1 be a pair of numbers such that $q \leq q_1 < \infty$ and $p_1 = (p-1)q_1$. Then $F \in H^{p_1}$ if and only if $k \in H^{q_1}$. For p an even integer, this contains the converse of Ryabykh's theorem, which was previously unknown. We also show that $F \in H^\infty$ if the coefficients of the Taylor expansion of k satisfy a certain growth condition.

Finally, we develop a method for finding explicit solutions to certain extremal problems in Bergman spaces. This method is applied to some particular classes of examples. Essentially the same method is used to study minimal interpolation problems, and it gives new information about canonical divisors in Bergman spaces.

CHAPTER I

Introduction

In this dissertation, we study extremal problems on Bergman spaces, which are certain spaces of analytic functions. Much of our work involves other spaces of analytic functions called Hardy spaces or H^p spaces.

We let $\mathbb{C}$ denote the complex numbers, and we let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. We let σ denote normalized Lebesgue area measure over $\mathbb{D}$, so that $\sigma(\mathbb{D}) = 1$. Then, for $0 < p < \infty$, the Bergman space $A^p(\mathbb{D})$, or simply A^p, consists of all functions analytic in the unit disc such that

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} |f|^p d\sigma \right\}^{1/p} < \infty.$$

In other words, $f \in A^p$ if f is analytic in the unit disc and is in L^p for Lebesgue area measure on the unit disc. We call $\|\cdot\|_{A^p}$ the A^p-norm; for $1 \leq p < \infty$ it is a true norm.

The Hardy spaces are closely related to the Bergman spaces. To define them, we need first to define the integral means of a function f analytic in $\mathbb{D}$. For $0 < p < \infty$ and $0 < r < 1$, the integral mean of f is

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

For $p = \infty$, we define $M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$. More generally, we could define the integral mean of a harmonic function in exactly the same way.

If f is a fixed analytic or harmonic function, and p is fixed, then $M_p(r, f)$ is an increasing function of r. For $0 < p \leq \infty$, we say that a function f is in the Hardy space H^p if f is analytic in $\mathbb{D}$ and $\|f\|_{H^p} = \lim_{r\to 1^-} M_p(r, f) < \infty$. We call $\|\cdot\|_{H^p}$ the H^p-norm; for $1 \leq p \leq \infty$ it defines a true norm. In a similar way, we define h^p to be the space of all real valued harmonic functions u in $\mathbb{D}$ such that $\lim_{r\to 1^-} M_p(r, u)$ is finite.

Note that $H^p \subset A^p$. Hardy spaces are generally more tractable than Bergman spaces, and all functions in Hardy spaces are well behaved in ways that some functions in Bergman spaces are not.

1.1 Basic Properties of Hardy Spaces

We now describe some basic facts about Hardy spaces for later reference. The Hardy space H^∞ is the space of bounded analytic functions in $\mathbb{D}$. The space H^2 is a Hilbert space, and the set $\{z^n\}_{n=0}^\infty$ is an orthonormal basis. If $p < q$, then $H^p \supset H^q$. For $1 \leq p \leq \infty$, the Hardy space H^p with norm $\|\cdot\|_{H^p}$ is a Banach space.

If $f \in H^p$ for $0 < p \leq \infty$, then its boundary function $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$, exists almost everywhere. In fact, each $f \in H^p$ has a nontangential limit at almost every point on the unit circle, although we will not need this fact. The boundary function of each $f \in H^p$ is in L^p and $\|f(e^{i\theta})\|_{L^p} = \|f\|_{H^p}$. If $f \in H^p$, for $0 < p < \infty$, it not only approaches its boundary values nontangentially, but "in the mean." In other words,

$$\lim_{r\to 1^-} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta \right\}^{1/p} = 0.$$

It follows that the polynomials are dense in H^p for $0 < p < \infty$. For an analytic function $f(z) = \sum_{j=0}^\infty a_j z^j$, let $S_n f(z) = \sum_{j=0}^n a_j z^j$ denote the n^{th} Taylor polynomial of f. If $f \in H^p$, then $S_n f \to f$ in H^p, where $1 < p < \infty$.

It is very useful to study the zero-sets of functions in the Hardy space. A basic tool for this is Jensen's formula. Let f be a function analytic when $|z| < \rho$ for some $\rho > 0$, and let $0 < r < \rho$. Suppose f has the zeros $z_1, z_2, \ldots$, repeated according to multiplicity, and that the Taylor series of f has leading term αz^m. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta = \sum_{|z_n|<r} \log\frac{r}{|z_n|} + \log(|\alpha| r^m).$$

Since the zeros of an analytic function are isolated, the set of z_n such that $|z_n| < r$ is finite, so there is no issue with convergence of the sum on the right hand side of the equation.

From Jensen's formula, one can show that if $f \in H^p$ for $0 < p \leq \infty$ and its zeros are $z_1, z_2, \ldots$ repeated according to multiplicity, then

$$\sum_n (1 - |z_n|) < \infty.$$

This is called the Blaschke condition.

Moreover, let $z_1, z_2, \ldots$ be nonzero complex numbers such that $\sum_n (1-|z_n|) < \infty$. Then we may form the infinite product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n}\frac{z_n - z}{1 - \overline{z_n} z},$$

which is called a Blaschke product. Such a product converges uniformly on compact subsets of the unit disc, and thus defines an analytic function. In fact, any Blaschke product is bounded in the unit disc and $|B(e^{i\theta})| = 1$ a.e. on the unit circle. The Blaschke product above has a zero of order m at the origin and has zeros at $z_1, z_2, \ldots$, and these are its only zeros.

For $0 < p \leq \infty$, any function $f \in H^p$ may be factored as $f = Bg$, where B is the Blaschke product formed from the zeros of f and g is non-vanishing. In this case, $\|f\|_{H^p} = \|g\|_{H^p}$.

We now discuss the *canonical factorization* on H^p. A function $f \in H^\infty$ is said to be an inner function if $|f(e^{i\theta})| = 1$ a.e. on the unit circle. Thus, every Blaschke product is an inner function. Let m denote Lebesgue measure on the unit circle. A singular inner function is a function of the form

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right),$$

where $\mu \perp m$. Each singular inner function is an inner function. If $\gamma \in \mathbb{R}$ and ψ is a function on the unit circle such that $\psi(t) \geq 0$ and $\log\psi \in L^1$, and $\psi(t) \in L^p$, then

$$F(z) = e^{i\gamma}\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log\psi(t)\,dt\right),$$

is called an outer function for H^p. Any such outer function is in fact in H^p. Also, $|F(e^{i\theta})| = \psi(e^{i\theta})$ a.e. on the unit circle. Each function $f \in H^p$ can be factored uniquely as $f = BSF$, where B is the Blaschke product formed from the zeros of f, and S is a singular inner function, and F is an outer function for H^p.

A very important fact about H^p functions is that if f and g are in H^p for some p, and $f(e^{i\theta}) = g(e^{i\theta})$ on some set of positive measure, then $f = g$. In particular, H^p functions are uniquely determined by their boundary values. Related to this is the fact that each H^p function can be identified with its boundary function, and that H^p can be identified with the subset of L^p consisting of all boundary functions of H^p functions. The isomorphism obtained from this identification is an isometry.

The dual space of H^p, denoted by $(H^p)^*$, consists of all continuous linear functionals from H^p to $\mathbb{C}$. For $1 \leq p < \infty$, the dual space $(H^p)^*$ is isometrically isomorphic to L^q/H^q, where $1/p + 1/q = 1$ and where an equivalence class $[g]$ of $g \in L^q$ corresponds to the functional defined by $f \mapsto (1/2\pi)\int_{\partial\mathbb{D}} f(z)g(z)\,dz$. Here $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, in other words the unit circle. Equivalently, we may identify $(H^p)^*$ with

L^q/H^q_0, where H^q_0 is the space of all $g \in H^q$ such that $g(0) = 0$, and an equivalence class $[g]$ for $g \in L^q$ corresponds to the functional $f \mapsto (1/2\pi)\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})\,d\theta$.

Alternatively, for $1 < p < \infty$, the dual space $(H^p)^*$ is isomorphic to H^q under the isomorphism for which $g \in H^q$ corresponds to the functional defined by $f \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta$. Note that strictly speaking, this isomorphism is not linear but conjugate-linear. Also, this isomorphism is not an isometry unless $p = 2$. However, for $1 < p < \infty$, there is a constant A_p depending only on p such that if $\phi \in (H^p)^*$ corresponds to $g \in H^q$, then

$$\|\phi\|_{(H^p)^*} \le \|g\|_{H^q} \le A_p\|\phi\|_{(H^p)^*}. \tag{1.1}$$

Closely related is the fact that the Szegő projection is bounded from L^p to H^p. The Szegő projection maps $L^1(\partial\mathbb{D})$ into the space of functions analytic in $\mathbb{D}$ and is defined by

$$(Sf)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{e^{it} - z} e^{it} dt.$$

If $f \in H^1$ then $Sf = f$, and if $f \in L^p$ for $1 < p < \infty$ and $f = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then $Sf(z) = \sum_{n=0}^{\infty} a_n z^n$. The fact that the Szegő projection is bounded from L^p onto H^p when $1 < p < \infty$ is equivalent to the theorem of M. Riesz that if a harmonic function $u \in h^p$ for some p with $1 < p < \infty$, and if v is the harmonic conjugate of u chosen so that $v(0) = 0$, then $v \in h^p$ and there is a constant B_p depending only on p such that $M_p(r, v) \le B_p M_p(r, u)$ for $0 \le r < 1$.

1.2 Basic Facts about Bergman Spaces

Recall that the space $A^p(\mathbb{D})$ is the space of all functions f analytic in the unit disc such that

$$\|f\|_{A^p} = \left\{\int_{\mathbb{D}} |f|^p\, d\sigma\right\}^{1/p} < \infty. \tag{1.2}$$

If $p < q$, then $A^p \supset A^q$. In fact, for any domain (i.e. open, connected set) in $\mathbb{C}$, we may define $A^p(\Omega)$ to be the space of all functions f analytic in Ω such that

$$\|f\|_{A^p(\Omega)} = \left\{ \int_\Omega |f|^p \, dA \right\}^{1/p} < \infty,$$

where dA denotes Lebesgue area measure. If Ω has at least one boundary component that consists of more than a single point, then one can show that $A^p(\Omega)$ is nonempty. For the unit disc, both definitions of A^p are the same, except that the norms differ by a constant multiple. When speaking of $A^p(\mathbb{D})$, we always use the definition in equation (1.2), since it simplifies matters by making $\|1\|_{A^p} = 1$.

As stated before, Bergman spaces are more difficult to work with than Hardy spaces. The first major difference is that a function in a Bergman space need not have boundary values. Also, while there is an analogue of canonical factorization for A^p functions, it does not give a unique factorization. These are just a few of the difficulties that arise in Bergman spaces that are absent in Hardy spaces.

It can be shown that point evaluation is a bounded linear functional in any A^p space. In fact, for a general domain Ω, we have that

$$|f(z)| \leq \pi^{-1/p} \delta(z)^{-2/p} \|f\|_{A^p(\Omega)}$$

for any function $f \in A^p(\Omega)$, where $\delta(z)$ is the distance from z to the boundary of Ω. Vukotić [25] obtained the estimate

$$|f(z)| \leq (1 - |z|^2)^{-2/p} \|f\|_{A^p(\mathbb{D})}$$

for any function $f \in A^p(\mathbb{D})$, where $0 < p < \infty$. For $p \geq 1$, this result was also obtained by Osipenko and Stessin [21]. The polynomials are dense in A^p for $0 < p < \infty$.

The simplest Bergman space is A^2, which is a Hilbert space. The polynomials $1, \sqrt{2}\, z, \sqrt{3}\, z^2, \ldots, \sqrt{n+1}\, z^n, \ldots$ form an orthonormal basis for A^2. Because point evaluation is a bounded linear functional on A^2, the space A^2 has a reproducing kernel $K(z,\zeta)$, called the *Bergman kernel*, with the property that

$$f(z) = \int_{\mathbb{D}} K(z,\zeta) f(\zeta)\, d\sigma(\zeta) \tag{1.3}$$

for all $f \in A^2$ and for all $z \in \mathbb{D}$. One can show that

$$K(z,\zeta) = \frac{1}{(1-\bar{\zeta} z)^2}.$$

Since the polynomials are dense in A^1, we have that (1.3) holds for all $f \in A^1$.

In fact, for any f in L^1 we many define the Bergman projection $\mathcal{P}$ by

$$(\mathcal{P}f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\bar{\zeta} z)^2}\, d\sigma(\zeta).$$

The Bergman projection maps L^1 into the space of functions analytic in $\mathbb{D}$. A non-trivial fact is that $\mathcal{P}$ also maps L^p boundedly onto A^p for $1 < p < \infty$. If $p = 2$, then $\mathcal{P}$ is just the orthogonal projection of L^2 onto A^2.

Closely related to the boundedness of the Bergman projection is the fact that, for $1 < p < \infty$, the dual space $(A^p)^*$ is isomorphic to A^q, where $1/p + 1/q = 1$. The isomorphism associates $k \in A^q$ with the functional defined by $f \mapsto \int_{\mathbb{D}} f\bar{k}\, d\sigma$. This isomorphism is not an isometry unless $p = 2$, but if $\phi \in (A^p)^*$ is represented by $k \in A^q$, then

$$\|\phi\| \le \|k\|_{A^q} \le C_p \|\phi\|, \tag{1.4}$$

where C_p is a constant depending only on p. Note that strictly speaking this isomorphism is not linear but conjugate linear.

We will often use equation (1.4) in this dissertation. Accordingly, we adopt the convention that q and p are conjugate exponents, unless otherwise specified.

If f is analytic in $\mathbb{D}$, recall that $S_n f$ denotes its n^{th} Taylor polynomial about the origin. Then for $1 < p < \infty$, if $f \in A^p$, the Taylor polynomials $S_n f \to f$ in A^p.

1.3 Extremal Problems in Hardy Spaces

Since this dissertation deals with the theory of extremal problems in Bergman spaces, we first review the corresponding theory for Hardy spaces. These problems were studied extensively around 1950 by S. Ya. Khavinson, and by W. W. Rogosinski and H. S. Shapiro (see [5], Chapter 8.) The basic problem is as follows: given a function $k \in L^q$ and the associated functional $\phi \in (H^p)^*$ defined by $\phi(f) = \int_{\partial\mathbb{D}} fk\,dz$, where $1/p+1/q = 1$, we wish to determine which functions $f \in H^p$ satisfy $\|f\|_{H^p} = 1$ and

$$\operatorname{Re}\phi(f) = \sup_{\|g\|_{H^p}=1} \operatorname{Re}\phi(g).$$

We will confine ourselves to discussion of the case where $1 < p < \infty$, although much of what we say carries over to the cases $p = 1$ or $p = \infty$.

In solving such problems it is helpful to consider the so called "dual extremal problem." If the functional ϕ is associated with $k \in L^q$, the dual extremal problem is to find a $K \in L^q$ such that $k - K \in H^q$ and

$$\|K\|_{L^q} = \min_{h \in H^q} \|k - h\|_{L^q}.$$

The function K is called the extremal kernel. It gives rise to the same functional as the original kernel.

By applying methods from functional analysis, one can show that, for $1 < p < \infty$, a unique solution always exists for a given extremal problem, and a unique solution always exists for the dual problem. In addition,

$$\max_{\|f\|_{H^p}=1} \operatorname{Re} \int_{\partial\mathbb{D}} fk\,dz = \min_{h \in H^q} \|k - h\|_{L^q}.$$

In other words, the $(H^p)^*$ norm of the functional ϕ equals the L^q norm of the extremal kernel K. We will generally let F denote the extremal function.

Using the conditions for equality in Hölder's inequality, one can show the following: Let $1 < p < \infty$, and let $k \in L^q$, where $1/p + 1/q = 1$, and define ϕ by $\phi(f) = \int_{\partial\mathbb{D}} fk\, dz$. Let $F \in H^p$ with $\|F\|_{H^p} = 1$ and $\phi(F) > 0$. Furthermore, let $K \in L^q$ be a function such that $k - K \in H^q$. Then a necessary and sufficient condition that F be the extremal function for k, and that K be the extremal kernel for k, is that both

1. $e^{i\theta}F(e^{i\theta})K(e^{i\theta}) \geq 0$ a.e., and
2. $|F(e^{i\theta})|^p = \|K\|_{H^q}^{-q}|K(e^{i\theta})|^q$ a.e.

Rational kernels are arguably the most important, since by the Cauchy integral formula they can represent linear combinations of any functionals which evaluate a function or one of its derivatives at a point in $\mathbb{D}$. More explicitly, for $z \in \mathbb{D}$ and $f \in H^p$, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\frac{1}{(\zeta - z)^{n+1}}\, d\zeta,$$

so $n!/(z-a)^{n+1}$ is the kernel for the functional defined by $f \mapsto f^{(n)}(a)$. For rational kernels, a structural formula exists for both F and K. Let the given kernel k be analytic in $\mathbb{D}$ except for poles at the points $\beta_1, \ldots, \beta_n$, listed according to multiplicity. Then there are numbers s and σ such that $0 \leq s \leq \sigma \leq n-1$, and numbers $a_1, \ldots a_{n-1}$, each of which lies in $\overline{\mathbb{D}}$, such that $a_1, \ldots, a_s$ are the zeros of K in $\mathbb{D}$, and $a_{s+1}, \ldots, a_\sigma$ are the zeros of F in $\mathbb{D}$, and $a_{\sigma+1}, \ldots, a_{n-1} \in \partial\mathbb{D}$, and such that

$$F(z) = A\prod_{j=s+1}^{\sigma}\frac{z-a_j}{1-\overline{a_j}z}\prod_{j=1}^{n-1}(1-\overline{a_j}z)^{2/p}\prod_{j=1}^{n}(1-\overline{\beta_j}z)^{-2/p}, \text{ and}$$

$$K(z) = B\prod_{j=1}^{s}\frac{z-a_j}{1-\overline{a_j}z}\prod_{j=1}^{n-1}(1-\overline{a_j}z)^{2/q}\prod_{j=1}^{n}\frac{(1-\overline{\beta_j}z)^{1-(2/q)}}{z-\beta_j},$$

where A and B are complex constants. Using these formulas, F and K can often be found explicitly.

Minimal interpolation problems are another important type of extremal problem. In these problems, we are given distinct points $z_1, z_2, \ldots, z_n$ in the unit disc, and nonnegative integers $d_1, d_2, \ldots, d_n$, and complex numbers $w_1, w_2, \ldots, w_n$, and we are required to find an $f \in H^p$ of smallest norm such that $f^{(d_j)}(z_j) = w_j$ for $1 \leq j \leq n$. One could also specify values at an infinite number of points, but we do not deal with this problem here.

Such a problem may in fact be formulated as a certain dual extremal problem, and thus for $1 < p < \infty$ the solution will always exist and be unique.

1.4 Previous Work on Extremal Problems in Bergman Spaces

Some previous work has been done on extremal problems in Bergman spaces, although the theory is nowhere near as complete as the H^p theory. Similarly to the Hardy space case, for a given non-zero functional $\phi \in (A^p)^*$, we study the extremal problem of finding a function $F \in A^p$ with norm $\|F\|_{A^p} = 1$ for which

$$\operatorname{Re} \phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re} \phi(g) = \|\phi\|. \tag{1.5}$$

Such a function F is called an extremal function.

This dissertation concentrates on the case when $1 < p < \infty$, and it is known in such a case that an extremal function always exists and is unique. In the A^1 case, there can be at most one extremal function, but one need not exist (see [26]). In the case where $1 < p < \infty$, we say that F is an extremal function for $k \in A^q$ if F solves the extremal problem for the functional ϕ with kernel k. Note that for $p = 2$, the extremal function is $F = k/\|k\|_{A^2}$.

We also deal with minimal interpolation problems on the Bergman space. An important example of a minimal interpolation problem is to find $F \in A^p$ such that F has specified zeros, $F(0) = 1$ (or $F^{(n+1)}(0) = 1$ if F is supposed to have n zeros at the origin), and $\|F\|_{A^p}$ is as small as possible. Essentially the same problem is to find $G \in A^p$ such that G has specified zeros, $\|G\|_{A^p} = 1$, and $\operatorname{Re}\{G(0)\}$ (or $\operatorname{Re}\{G^{(n+1)}(0)\}$

if G is supposed to have n zeros at the origin) is as large as possible.

For $0 < p < \infty$, such a function G exists and is unique, as long as the specified zero set is actually the zero set of some A^p function. Such a G is called a contractive zero-divisor or canonical divisor, and it has several remarkable properties. For instance, it has no zeros other than those specified. Another property is that if $f \in A^p$ and every zero of G is also a zero of f, then $f/G \in A^p$ and in fact $\|f/G\|_{A^p} \leq \|f\|_{A^p}$. These contractive divisors play a similar role to Blaschke products in the H^p theory, although Blaschke products are isometric divisors in H^p, since $\|f/B\|_{H^p} = \|f\|_{H^p}$ when f vanishes on the zero-set of B. In fact, canonical divisors were first discovered in $p = 2$, by Hedenmalm [13]. He noted that Blaschke products are solutions to the extremal problem of finding the function $f \in H^2$ with norm 1, specified zeros, and the largest possible value at the origin (or the largest possible value of the n^{th} derivative if f is required to have n zeros at the origin). Duren, Khavinson, Shapiro, and Sundberg extended his results to $0 < p < \infty$ in a series of papers (see [8] and [7]). MacGregor and Stessin have obtained a structural formula for canonical divisors with a finite zero set in [20].

Before canonical divisors were discovered, Horowitz found functions with some of the same properties as canonical divisors, though his functions were not necessarily in A^p. He used them to obtain important results on zero-sets of A^p functions (see [16], [15], and [9]).

Vukotić [26] summarizes some known results, as well as discussing the A^1 case in more detail than we do here. In [25], he completely solves the extremal problem for point evaluation functionals. His result was also obtained, in less general form, by Osipenko and Stessin in [21].

Ryabykh [22] obtained an important result about extremal problems in Bergman spaces with kernels in Hardy spaces, a topic which we study in this dissertation. Khavinson and Stessin have obtained results about polynomial or rational kernels for $1 < p < \infty$ in [18]. Extremal problems in Bergman space are studied in many other places. See for example [1] and [17].

1.5 Two Basic Tools

We will use an important characterization of extremal functions in closed subspaces of L^p for $1 < p < \infty$ (see [23], p. 55).

Theorem A. *Let $1 < p < \infty$, let X be a closed subspace of L^p, and let $\phi \in X^*$. Assume that ϕ is not identically 0. A function $F \in X$ with $\|F\| = 1$ satisfies*

$$\operatorname{Re} \phi(F) = \sup_{g \in X, \|g\|=1} \operatorname{Re} \phi(g) = \|\phi\|_{X^*}$$

if and only if $\phi(F) > 0$ and

$$\int_{\mathbb{D}} h |F|^{p-1} \overline{\operatorname{sgn} F} \, d\sigma = 0$$

for all $h \in X$ with $\phi(h) = 0$. If F satisfies the above conditions, then

$$\int_{\mathbb{D}} h |F|^{p-1} \overline{\operatorname{sgn} F} \, d\sigma = \frac{\phi(h)}{\|\phi\|_{X^*}}$$

for all $h \in X$.

For the sake of completeness, we will provide a proof.

Proof. Let q be the conjugate exponent to p, so that $1/p + 1/q = 1$. By the Hahn-Banach theorem, there is a functional $\Phi \in (L^p)^*$ such that $\|\Phi\| = \|\phi\|$ and such that ϕ is the restriction of Φ to X. Let $k \in L^q$ be the integral kernel of Φ, which by the Riesz representation theorem exists and has the property that $\|k\|_{L^q} = \|\Phi\|$.

First, suppose that $F \in X$ with $\|F\| = 1$ and

$$\operatorname{Re} \phi(F) = \sup_{g \in X, \|g\|=1} \operatorname{Re} \phi(g) = \|\phi\|.$$

Then

$$\|k\|_{L^q} = \|\Phi\| = \Phi(F) = \int_{\mathbb{D}} Fk\, d\sigma \le \|k\|_{L^q} \|F\|_{L^q} = \|k\|_{L^q}.$$

So by the conditions for equality in Hölder's inequality, $k = \|k\|_{L^q} |F|^{p-1} \overline{\operatorname{sgn} F}$, so $\int_{\mathbb{D}} |F|^{p-1} \overline{\operatorname{sgn} F} h\, d\sigma = 0$ for all $h \in X$ such that $\phi(h) = 0$.

Conversely, suppose that $\|F\| = 1$ and $\int_{\mathbb{D}} |F|^{p-1} \overline{\operatorname{sgn} F} h\, d\sigma = 0$ for all $h \in X$ such that $\phi(h) = 0$. Then for any $f \in X$, with $\|f\| = 1$, we have $\phi[f - \phi(f)(F/\phi(F))] = 0$ so

$$\begin{aligned} 0 = \phi\left(f - \phi(f)\frac{F}{\phi(F)}\right) &= \int_{\mathbb{D}} \left[f - \phi(f)\frac{F}{\phi(F)}\right] |F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma \\ &= \int_{\mathbb{D}} f|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma - \int_{\mathbb{D}} \frac{\phi(f)}{\phi(F)} |F|^p\, d\sigma \\ &= \int_{\mathbb{D}} f|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma - \frac{\phi(f)}{\phi(F)}. \end{aligned}$$

But $\|F\| = 1$, so by Hölder's inequality,

$$\frac{|\phi(f)|}{\phi(F)} = \left|\int_{\mathbb{D}} f|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma\right| \le \|f\|_{L^p} \||F|^{p-1}\overline{\operatorname{sgn} F}\|_{L^q} = \|f\|_{L^p} = 1.$$

Thus, $\operatorname{Re} \phi(f) \le \operatorname{Re} \phi(F)$, since $\phi(F) > 0$. This finishes the proof of the first part of the theorem.

For the second part, note that if F has the extremal property in question, then for any $h \in X$, we have that

$$\begin{aligned} \int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F} d\sigma &= \int_{\mathbb{D}} \left[\left(h - \phi(h)\frac{F}{\phi(F)}\right) + \phi(h)\frac{F}{\phi(F)}\right] |F|^{p-1} \operatorname{sgn} \overline{F}\, d\sigma \\ &= 0 + \frac{\phi(h)}{\phi(F)}\|F\| = \frac{\phi(h)}{\|\phi\|_{X^*}} \end{aligned}$$

since $\phi(h - (\phi(h)/\phi(F))F) = 0$ and $\|F\| = 1$. $\square$

The fact that if F is extremal then $\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = 0$ for all h such that $\phi(h) = 0$ may also be proven by a variational method (see [9], p.123).

Similar methods to those above may be used to prove the following theorem. It may be found in [23], p. 55.

Theorem B. *Suppose that X is a closed subspace of $L^p(\mathbb{D})$. Let $F \in L^p$ and suppose that for all $h \in X$, we have $\|F\| \le \|F + h\|$. Then*

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = 0$$

for all $h \in X$.

We will also make repeated use of the Cauchy-Green theorem. Recall that $\partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$ and that $\partial/\partial \overline{z} = (1/2)(\partial/\partial x + i\partial/\partial y)$. Also, $dz = dx + idy$ and $d\overline{z} = dx - idy$.

Cauchy-Green Theorem. *If Ω is a region in the plane with piecewise smooth boundary and $f \in C^1(\overline{\Omega})$, then*

$$\frac{1}{2i}\int_{\partial\Omega} f(z)\, dz = \int_{\Omega} \frac{\partial}{\partial \overline{z}} f(z)\, dA(z),$$

where $\partial\Omega$ denotes the boundary of Ω.

We will often use this theorem in its "conjugate" version, which says that

$$\frac{i}{2}\int_{\partial\Omega} f(z)\, d\overline{z} = \int_{\Omega} \frac{\partial}{\partial z} f(z)\, dA(z)$$

as long as the conditions in the theorem hold. This can be derived from the Cauchy-Green theorem by replacing f by $\overline{f}$ and then taking complex conjugates of both sides of the resulting equation.

1.6 Summary of Results

In Chapter II, we define the notion of uniform convexity, and use it to show that the extremal function F and kernel k depend continuously on each other in A^p spaces for $1 < p < \infty$. We then use the results of that chapter to give a simplified proof of Ryabykh's Theorem, which states that for A^p-extremal problems, if the kernel k is in the Hardy space H^q, then the extremal function F is in H^p. This chapter first appeared, in modified form, in [10].

In Chapter III, we extend Ryabykh's Theorem in the case where p is an even integer. We show that if $k \in H^{q_1}$ for some q_1 such that $q \leq q_1 < \infty$, then $F \in H^{(p-1)q_1}$. We also prove the converse to this statement, and show that if the Fourier coefficients of k are sufficiently small, then $F \in H^\infty$.

In Chapter IV, we develop techniques to solve certain extremal problems explicitly. We also give a characterization of canonical divisors.

CHAPTER II

Uniformly Convex Spaces and Ryabykh's Theorem

We begin this section by studying extremal problems over uniformly convex Banach spaces, a type of Banach space that includes each of L^p, H^p, and A^p when $1 < p < \infty$. Given a uniformly convex Banach space X and a linear functional $\phi \in X^*$, we ask the question: what elements x of the space with norm $\|x\| = 1$ maximize $\operatorname{Re} \phi(x)$? Because of the uniform convexity, this problem will always have a unique solution, which is called the extremal element. In this chapter, we show that the extremal element depends continuously on the functional ϕ, and it can be approximated by the solutions of the same extremal problem over subspaces of the original Banach space. Using these results, we give a streamlined proof of a theorem of Ryabykh, which says that for a functional defined on the Bergman space A^p, with kernel in the Hardy space H^q, the extremal element is in H^p, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2.1 Uniform convexity and extremal problems

Let X be a complex Banach space and let X^* be its dual space. For a given linear functional $\phi \in X^*$ with $\phi \neq 0$, we are interested in all elements $x \in X$ with norm

$\|x\| = 1$ such that

$$\operatorname{Re}\phi(x) = \sup_{\|y\|=1} \operatorname{Re}\phi(y) = \|\phi\|. \tag{2.1}$$

As stated in the introduction, such a problem is referred to as an extremal problem, and x is an extremal element.

A closely related problem is that of finding $x \in X$ such that

$$\phi(x) = 1 \qquad \text{and} \qquad \|x\| = \inf_{\phi(y)=1} \|y\|. \tag{2.2}$$

If x solves the problem (2.1), then $\frac{x}{\phi(x)}$ solves the problem (2.2), and if x solves (2.2), then $\frac{x}{\|x\|}$ solves (2.1). To standardize notation, when dealing with elements of general uniformly convex Banach spaces, we will often denote solutions to (2.1) by $x^\star$ and solutions to (2.2) by $x^\diamond$.

For general Banach spaces, the problem (2.1) need not have a solution, and if it does the solution need not be unique. However, if the Banach space X is *uniformly convex*, there will always be a unique solution.

Definition 2.1. A Banach space X is said to be uniformly convex if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$\left\|\tfrac{1}{2}(x+y)\right\| > 1 - \delta \qquad \text{implies} \qquad \|x - y\| < \varepsilon.$$

An equivalent statement is that if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$. Uniform convexity was introduced by Clarkson [3], who proved that the L^p spaces are uniformly convex for $1 < p < \infty$.

Proposition 2.2. *Let X be a uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X. If for some $d > 0$, $\|x_n\| \to d$ and $\|y_n\| \to d$ as $n \to \infty$, and $\|x_n + y_n\| \to 2d$, then $\|x_n - y_n\| \to 0$.*

Proof. We have that

$$\begin{aligned} 2 \geq \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| &= \frac{1}{\|x_n\|} \left\| x_n + \frac{\|x_n\|}{\|y_n\|} y_n \right\| \\ &\geq \frac{1}{\|x_n\|} \|x_n + y_n\| - \frac{1}{\|x_n\|} \left\| y_n - \frac{\|x_n\|}{\|y_n\|} y_n \right\| \to 2 \end{aligned}$$

as $n \to \infty$. Hence by uniform convexity,

$$\left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| \to 0.$$

But then

$$\begin{aligned} \|x_n - y_n\| &= \|x_n\| \left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|} \right\| \\ &\leq \|x_n\| \left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| + \|x_n\| \left\| y_n \left(\frac{1}{\|x_n\|} - \frac{1}{\|y_n\|} \right) \right\| \to 0. \end{aligned}$$

□

The following result is basic (see for instance [9], Section 2.2) and gives immediately the existence and uniqueness of extremal elements.

Proposition 2.3. *A closed convex subset of a uniformly convex Banach space has exactly one element of smallest norm.*

Since the problem (2.2) is one of finding an element of minimal norm over a (closed) subspace of the Banach space, it has a unique solution, and thus we obtain the following theorem.

Theorem 2.4. *If X is a uniformly convex Banach space and $\phi \in X^*$ with $\phi \neq 0$, then the problems* (2.1) *and* (2.2) *both have a unique solution.*

For later reference, we record the relations

$$\begin{aligned} x^\star &= \frac{x^\diamond}{\|x^\diamond\|} = \|\phi\| x^\diamond, \\ x^\diamond &= \frac{x^\star}{\phi(x^\star)} = \frac{x^\star}{\|\phi\|}, \\ \|\phi\| &= \phi(x^\star) = \frac{1}{\|x^\diamond\|}. \end{aligned} \tag{2.3}$$

We now state a lemma which will be applied repeatedly.

Lemma 2.5. *Let X be a uniformly convex Banach space, let $\{x_n\}$ and $\{y_n\}$ be sequences in X, and let $\phi \in X^*$ where $\phi \neq 0$. If for some $d \geq 0$, $\|x_n\| \to d$ and $\|y_n\| \to d$ as $n \to \infty$, and if $|\phi(x_n + y_n)| \to 2d\|\phi\|$, then $\|x_n - y_n\| \to 0$.*

Proof. Since $|\phi(x_n + y_n)| \leq \|\phi\|\|x_n + y_n\|$, we have that

$$\frac{|\phi(x_n + y_n)|}{\|\phi\|} \leq \|x_n + y_n\| \leq \|x_n\| + \|y_n\|.$$

But the left and right sides of this inequality both approach $2d$, so Proposition 2.2 gives the result. □

2.2 Continuous dependence of the solution on the functional

It is important to know whether the extremal element depends continuously on the functional. This turns out to be true for uniformly convex spaces. Note that when say a sequence of linear functionals ϕ_n approaches a linear functional ϕ, we mean that $\|\phi - \phi_n\| \to 0$.

Theorem 2.6. *Suppose that X is a uniformly convex Banach space and that $\{\phi_n\}$ is a sequence of nonzero functionals in X^* such that $\phi_n \to \phi \neq 0$. Let $x_n^\star$ denote the solution to problem (2.1) for ϕ_n, and let $x^\star$ be the solution for ϕ. Similarly, let $x_n^\diamond$ denote the solution to problem (2.2) for ϕ_n, and let $x^\diamond$ be the solution for ϕ. Then $x_n^\star \to x^\star$ and $x_n^\diamond \to x^\diamond$.*

Ryabykh [22] gives a different proof of this statement while establishing another theorem, but our proof is simpler and more direct.

Proof. Note that

$$\phi(x_n^\star) = \phi_n(x_n^\star) + (\phi - \phi_n)(x_n^\star) = \|\phi_n\| + (\phi - \phi_n)(x_n^\star) \to \|\phi\|$$

so that

$$\phi(x_n^\star + x^\star) \to 2\|\phi\|.$$

Lemma 2.5 now shows that $x_n^\star \to x^\star$. It follows that $x_n^\diamond \to x^\diamond$ since

$$x_n^\diamond = \frac{x_n^\star}{\|\phi_n\|} \to \frac{x^\star}{\|\phi\|} = x^\diamond.$$

$\square$

For given ϕ, a unique $x^\diamond$ solves the problem (2.2):

$$\|x^\diamond\| = \min_{\phi(x)=1} \|x\|.$$

It is natural to ask whether different functionals can give rise to the same solution of the problem (2.2). The following theorem answers this question when X^* is uniformly convex.

Theorem 2.7. *Let X be a Banach space and let $x \in X$ with $x \neq 0$. If X^* is uniformly convex, then there exists a unique $\phi \in X^*$ such that x solves the problem* (2.2) *associated with ϕ.*

Proof. By the Hahn-Banach theorem, there is some $\phi \in X^*$ such that $\phi(x) = 1$ and $\|\phi\| = \frac{1}{\|x\|}$. But if for some $y \in X$, $\phi(y) = 1$, then $1 \leq \|\phi\|\|y\|$, or $\|y\| \geq \|\phi\|^{-1} = \|x\|$. This says that x solves the problem (2.2) associated with ϕ. To show that ϕ is unique, consider the problem of finding $\psi^\diamond$ such that

$$\|\psi^\diamond\| = \min_{\psi \in X^*, \psi(x)=1} \|\psi\| \tag{2.4}$$

We claim that if x solves the problem (2.2) for some $\theta \in X^*$, then θ solves the problem (2.4). In particular, ϕ solves the problem (2.4). To see this, note that if x solves (2.2) for θ, then $\theta(x) = 1$. If θ is not a solution of (2.4), then there is a functional ψ such that $\|\psi\| < \|\theta\|$ and $\psi(x) = 1$. But this is impossible, since it would imply

$$1 = |\psi(x)| \leq \|\psi\|\|x\| = \frac{\|\psi\|}{\|\theta\|} < 1,$$

where we have used the last relation in (2.3). Since X^* is uniformly convex, Theorem 2.4 shows that ϕ is the unique solution to (2.4), which proves the theorem. □

When $x^\diamond$ determines the functional ϕ uniquely, it is also natural to ask whether ϕ depends continuously on $x^\diamond$. The following theorem answers this question when X^* is uniformly convex.

Theorem 2.8. *(a) Suppose that X is a Banach space whose dual space X^* is uniformly convex. If S is a closed subspace of X, then for any $x \in S$, there exists a unique $\phi \in S^*$ such that x solves the problem* (2.2) *associated with ϕ over S.*

(b) Moreover, if $x_n \in S$ and $x_n \to x$, and ϕ_n is the unique functional in S^* that solves the problem (2.2) *for x_n, then $\phi_n \to \phi$.*

Proof. Recall that if S is a closed subspace of X, then S^* is isometrically isomorphic to $X^*/S^\perp$, where $S^\perp$ is the annihilator of S in X^*. In [19], Section 26, it is shown that the quotient space of a uniformly convex space is uniformly convex, which shows that S^* is uniformly convex. From this and Theorem 2.7, part (a) follows.

Since, as shown in the proof of Theorem 2.7, each ϕ_n is the unique solution to the problem (2.4) with x_n in place of x, and since ϕ is the unique solution of the problem (2.4), Theorem 2.6 implies part (b).

□

Since $(L^p)^* = L^q$ is uniformly convex for $1 < p < \infty$, this theorem applies to the spaces A^p and H^p for $1 < p < \infty$. Note that if $\|x\| = 1$, there is not a unique linear functional ϕ such that x solves problem (2.1) for ϕ. Indeed, if x solves (2.1) for ϕ, then it also does so for any positive scalar multiple of ϕ. However, we can say something about uniqueness of ϕ in this case. If $\|x\| = 1$, then x solves problem (2.2) for ϕ if and only if x solves problem (2.1) for ϕ, as follows from the relations (2.3). Thus, if $\|x\| = 1$, then there is a ϕ such that x solves (2.1) for ϕ, and this ϕ is unique up to positive scalar multiple.

2.3 Approximation by solutions in subspaces

To obtain Ryabykh's theorem, we will also need the following theorem, which allows an extremal element to be approximated by extremal elements over subspaces.

Theorem 2.9. *Suppose that X is a uniformly convex Banach space and let X_1, $X_2, X_3, \ldots$ be (closed) subspaces for which $X_1 \subset X_2 \subset \cdots \subset X$ and*

$$\overline{\bigcup_{n\in\mathbb{N}} X_n} = X.$$

Let $\phi \in X^*$, and let

$$\|\phi\|_n = \sup_{x\in X_n, \|x\|=1} |\phi(x)|.$$

Let $x_n^\star$ denote the solution to the problem (2.1) when restricted to the subspace X_n, and let $x_n^\diamond$ denote the solution to the problem (2.2) when restricted to X_n. Then $\|\phi\|_n \to \|\phi\|$ and $x_n^\star \to x^\star$ and $x_n^\diamond \to x^\diamond$ as $n \to \infty$.

Here, $x^\star$ denotes the solution to (2.1) over X, and $x^\diamond$ denotes the solution to (2.2) over X.

Proof. First of all, we know that each $x_n^\star$ and $x_n^\diamond$ is uniquely determined since a closed subspace of a uniformly convex space is uniformly convex. Let $\varepsilon > 0$ be given. Since $\bigcup_{n\in N} X_n$ is dense in X, we may choose an n such that $\|x^\star - y\| < \varepsilon$ for some $y \in X_n$.

Thus

$$|\phi(y)| = |\phi(x^\star) - \phi(x^\star - y)| \ge |\phi(x^\star)| - |\phi(x^\star - y)| = \|\phi\| - |\phi(x^\star - y)|$$

$$\ge \|\phi\| - \|\phi\|\,\|x^\star - y\| \ge \|\phi\|(1-\varepsilon).$$

We also know that $\|y\| \le 1+\varepsilon$, so

$$\|\phi\|_n \ge \frac{|\phi(y)|}{\|y\|} \ge \frac{(1-\varepsilon)\|\phi\|}{1+\varepsilon},$$

and thus for all $N \ge n$,

$$\|\phi\|_N \ge \frac{(1-\varepsilon)\|\phi\|}{1+\varepsilon}.$$

But since $\|\phi\| \ge \|\phi\|_m$ for all m, this implies that

$$\|\phi\| \ge \limsup_{m\to\infty} \|\phi\|_m \ge \liminf_{m\to\infty} \|\phi\|_m \ge \frac{(1-\varepsilon)\|\phi\|}{1+\varepsilon}.$$

Because ε was arbitrary, this shows that $\|\phi\|_m \to \|\phi\|$.

Now, $\phi(x_n^\star + x^\star) = \|\phi\|_n + \|\phi\| \to 2\|\phi\|$, so Lemma 2.5 shows that $\|x^\star - x_n^\star\| \to 0$. For $x^\diamond$, the result now follows since

$$x_n^\diamond = \frac{x^\star}{\|\phi\|_n} \qquad \text{and} \qquad x^\diamond = \frac{x^\star}{\|\phi\|}.$$

$\square$

2.4 Ryabykh's Theorem

With the help of the preceding results, we can now obtain a slightly sharpened version of Ryabykh's theorem. Our proof uses some of Ryabykh's ideas but is simpler and more concise. We note that Ryabykh's approach successfully applies to other extremal problems as well. For example, it applies to some problems of best approximation of a given function by harmonic or analytic functions (see [17]), and to some nonlinear extremal problems involving non-vanishing functions (see [1]). Note that we let F denote the solution to the extremal problem instead of $f^\star$, for typographical convenience.

Theorem 2.10. *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Suppose that $\phi \in (A^p)^*$ and $\phi(f) = \int_{\mathbb{D}} f\overline{k}\, d\sigma$ for some $k \in H^q$, where $k \neq 0$. Then the solution to the extremal problem* (2.1) *(with $X = A^p$) belongs to H^p and satisfies*

$$\|F\|_{H^p} \leq \left\{ \left[\max(p-1,1) \right] \frac{C_p \|k\|_{H^q}}{\|k\|_{A^q}} \right\}^{1/(p-1)}, \tag{2.5}$$

where C_p is the constant in (1.4)*.*

Of course, this implies that the solution to the problem (2.2) is in H^p as well.

Proof. Let

$$K(z) = \frac{1}{z} \int_0^z k(\zeta)\, d\zeta,$$

so that $(zK)' = k$.

Now, the continuous form of Minkowski's inequality gives

$$\begin{aligned} M_q(r, zK) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |re^{i\theta} K(re^{i\theta})|^q \, d\theta \right\}^{1/q} &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^r k(\rho e^{i\theta}) e^{i\theta}\, d\rho \right|^q \, d\theta \right\}^{1/q} \\ \leq \int_0^r \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k(\rho e^{i\theta})|^q \, d\theta \right\}^{1/q} d\rho &\leq \int_0^r M_q(k, \rho)\, d\rho \leq \|k\|_{H^q}. \end{aligned}$$

Thus,

$$|K\|_{H^q} \leq \|k\|_{H^q}. \tag{2.6}$$

To facilitate calculations involving the Cauchy-Green theorem, we suppose first that $k \in C^1(\overline{\mathbb{D}})$. Let F_n denote the solution to the extremal problem (2.1) over the space of all polynomials of degree n or less, considered as a subspace of A^p, and let F be the solution to the same problem over the space A^p. Then by the Cauchy-Green theorem,

$$\|F_n\|_{H^p}^p = \frac{1}{2\pi} \int_0^{2\pi} |F_n(e^{i\theta})|^p \, d\theta = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} |F_n(z)|^p \, \overline{z}\, dz$$

$$= \int_{\mathbb{D}} \left(F_n + \frac{p}{2} z F_n{}' \right) |F_n|^{p-1} \overline{\operatorname{sgn} F_n}\, d\sigma.$$

Because $(F_n + \frac{p}{2} z F_n{}')$ is a polynomial of degree at most n, we can appeal to Theorem A with X taken to be the subspace of A^p consisting of all such polynomials. The theorem shows that

$$\begin{aligned}\|F_n\|_{H^p}^p &= \frac{1}{\|\phi\|_n} \phi\left(F_n + \frac{p}{2} z F_n{}'\right) = \frac{1}{\|\phi\|_n} \int_{\mathbb{D}} \overline{k} \left(F_n + \frac{p}{2} z F_n{}'\right) d\sigma \\ &= \frac{1}{\|\phi\|_n} \int_{\mathbb{D}} \left[\frac{\partial}{\partial \overline{z}} \left(F_n \overline{zK}\right) + \frac{p}{2} \left(\frac{\partial}{\partial z} \left(z F_n \overline{k}\right) - \frac{\partial}{\partial \overline{z}} \left(F_n \overline{zK}\right)\right)\right] d\sigma.\end{aligned}$$

Now another application of the Cauchy-Green theorem gives:

$$\begin{aligned}\|F_n\|_{H^p}^p &= \frac{1}{2\pi i \|\phi\|_n} \int_{\partial\mathbb{D}} \overline{zK} F_n \, dz - \frac{p}{4\pi i \|\phi\|_n} \int_{\partial\mathbb{D}} z F_n \overline{k} \, d\overline{z} - \frac{p}{4\pi i \|\phi\|_n} \int_{\partial\mathbb{D}} F_n \overline{zK} \, dz \\ &= \frac{1}{2\pi \|\phi\|_n} \int_0^{2\pi} F_n \left[\left(\frac{p}{2}\right) \overline{k} + \left(1 - \frac{p}{2}\right) \overline{K} \right] d\theta \\ &\le \frac{1}{\|\phi\|_n} \|F_n\|_{H^p} \left\| \left(\frac{p}{2}\right) k + \left(1 - \frac{p}{2}\right) K \right\|_{H^q}\end{aligned}$$

Minkowski's inequality now gives

$$\|F_n\|_{H^p}^p \le \frac{1}{\|\phi\|_n} \|F_n\|_{H^p} \left(\frac{p}{2} \|k\|_{H^q} + \left|1 - \frac{p}{2}\right| \|K\|_{H^q} \right).$$

But this implies that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F_n(re^{i\theta})|^p \, d\theta \right)^{1/p} \le \frac{1}{\|\phi\|_n^{1/(p-1)}} \left(\frac{p}{2} \|k\|_{H^q} + \left|1 - \frac{p}{2}\right| \|K\|_{H^q} \right)^{1/(p-1)},$$

when $0 < r < 1$. From Theorem 2.9 and the fact that convergence in A^p implies uniform convergence on compact subsets of the disc, it follows that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p} \le \frac{1}{\|\phi\|^{1/(p-1)}} \left(\frac{p}{2} \|k\|_{H^q} + \left|1 - \frac{p}{2}\right| \|K\|_{H^q} \right)^{1/(p-1)},$$

when $0 < r < 1$. Now we apply (2.6) to infer that

$$\|F\|_{H^p} \le \left\{ \left(\frac{p}{2} + \left|1 - \frac{p}{2}\right| \right) \frac{\|k\|_{H^q}}{\|\phi\|} \right\}^{1/(p-1)}.$$

Since we know that $\|k\|_{A^q} \le C_p \|\phi\|$, and that $p/2 + |1 - (p/2)| = \max(p-1, 1)$, we conclude finally that the inequality (2.5) holds under the assumption that $k \in C^1(\overline{\mathbb{D}})$.

If k is a general function in H^q, we can approximate it in H^q norm by a sequence of functions $k_m \in C^1(\overline{\mathbb{D}})$. (We may even use polynomials, by [5], Theorem 3.3.) Then the corresponding functionals ϕ_m converge to ϕ, so by Theorem 2.6, the extremal elements F_m for ϕ_m converge to the extremal element F for ϕ in A^p norm. Since $k_m \in C^1(\overline{\mathbb{D}})$, we have already found that

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |F_m(re^{i\theta})|^p \, d\theta\right)^{1/p} \leq \left\{\left[\max(p-1,1)\right]\frac{C_p\|k_m\|_{H^q}}{\|k_m\|_{A^q}}\right\}^{1/(p-1)}, \qquad 0<r<1.$$

But the convergence of $F_m \to F$ in A^p norm implies that $F_m(z) \to F(z)$ locally uniformly, so it follows that

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta\right)^{1/p} \leq \left\{\left[\max(p-1,1)\right]\frac{C_p\|k\|_{H^q}}{\|k\|_{A^q}}\right\}^{q/p}, \qquad 0<r<1,$$

which proves (2.5). $\square$

CHAPTER III

Extensions of Ryabykh's Theorem

In this chapter, we obtain a sharper version of Ryabykh's theorem in the case where p is an even integer. As before, we will let the function $k \in A^q$ be the kernel of a functional in $(A^p)^*$, and $F \in A^p$ will be the corresponding extremal function. Our results are:

- For $q \leq q_1 < \infty$, the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$.
- If the Taylor coefficients of k are "small enough," then $F \in H^\infty$.
- The map sending a kernel $k \in H^q$ to its extremal function $F \in A^p$ is a continuous map from $H^q \setminus 0$ into H^p.

Our proofs rely heavily on Littlewood-Paley theory, and seem to require that p be an even integer. It is an open problem whether the results hold without this assumption.

To obtain our results, including a generalization of Ryabykh's theorem, we will need the following technical lemmas. Their proofs, which involve Littlewood-Paley theory, are deferred to the end of the chapter.

In the statement of the lemmas, we will use the following definition of principal value. If h is a measurable function in the unit disc, define the principal value of its integral as

$$\text{p. v.} \int_{\mathbb{D}} h\, dA = \lim_{r\to 1} \int_{r\mathbb{D}} h\, dA,$$

if the limit exists. Also, recall that $S_n f$ denotes the n^{th} Taylor polynomial of f.

Lemma 3.1. *Let p be an even integer. Let $f \in H^p$ and let h be a polynomial. Then*

$$\text{p. v.} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} f' h\, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} (S_n f)' h\, d\sigma.$$

Lemma 3.2. *Suppose that $1 < p_1 < \infty$ and $1 < p_2, p_3 \le \infty$, and also that*

$$1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Let $f_1 \in H^{p_1}$, $f_2 \in H^{p_2}$, and $f_3 \in H^{p_3}$. Then

$$\left| \text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3'\, d\sigma \right| \le C \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}}$$

where C depends only on p_1 and p_2. (Implicit is the claim that the principal value exists.) Moreover, if $p_3 < \infty$, then

$$\text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3'\, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} \overline{f_1} f_2 (S_n f_3)'\, d\sigma.$$

3.1 The Norm-Equality

Let p be an even integer and let q be its conjugate exponent. Let $k \in H^q$ and let F be the extremal function for k over A^p. We will denote by ϕ the functional associated with k. Let F_n be the extremal function for k when the extremal problem is posed over P_n, the space of polynomials of degree at most n. Also, let

$$K(z) = \frac{1}{z} \int_0^z k(\zeta)\, d\zeta, \tag{3.1}$$

so that $(zK)' = k$. During proof of Ryabykh's theorem in Chapter II, an important step was to show that

$$\frac{1}{2\pi} \int_0^{2\pi} |F_n(e^{i\theta})|^p\, d\theta = \frac{1}{2\pi \|\phi_{|P_n}\|} \int_0^{2\pi} F_n \left[\left(\frac{p}{2}\right) \overline{k} + \left(1 - \frac{p}{2}\right) \overline{K} \right] d\theta.$$

We will now derive a similar result for F:

Theorem 3.3. *Let p be an even integer, let $k \in H^q$, and let $F \in A^p$ be the extremal function for k. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta})\,d\theta = \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F\left[\left(\frac{p}{2}\right) h\overline{k} + \left(1-\frac{p}{2}\right)(zh)'\overline{K}\right] d\theta,$$

for every polynomial h.

Proof. Since Ryabykh's theorem says that $F \in H^p$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta})\,d\theta = \lim_{r\to 1}\frac{i}{2\pi}\int_{\partial(r\mathbb{D})} |F(z)|^p h(z) z\,d\overline{z},$$

where h is any polynomial. Apply the Cauchy-Green theorem to transform the right-hand side into

$$\text{p. v.}\ \frac{1}{\pi}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zhF'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\,dA(z).$$

Invoking Lemma 3.1 with zh in place of h shows that this limit equals

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zh(S_nF)'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\,dA(z).$$

Since $(zh)'F + \frac{p}{2}zh(S_nF)'$ is in A^p, we may apply Theorem A to reduce the last expression to

$$\lim_{n\to\infty}\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zh(S_nF)'\right)\overline{k}\,dA(z). \tag{3.2}$$

Recall that we have defined $K(z) = \frac{1}{z}\int_0^z k(\zeta)\,d\zeta$. To prepare for a reverse application of the Cauchy-Green theorem, we rewrite the integral in (3.2) as

$$\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}}\Bigg[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\} + \frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\} - \frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\Bigg] dA(z).$$

Now this equals

$$\lim_{r\to 1}\frac{1}{\pi\|\phi\|}\int_{r\mathbb{D}}\left[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\}+\frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\}\right.$$
$$\left.-\frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\right]dA(z).$$

We apply the Cauchy-Green theorem to show that this equals

$$\lim_{r\to 1}\left[\frac{1}{2\pi i\|\phi\|}\int_{\partial(r\mathbb{D})}(zh)'F\overline{zK}\,dz+\frac{ip}{4\pi\|\phi\|}\int_{\partial(r\mathbb{D})}zhS_n(F)\overline{k}\,d\overline{z}\right.$$
$$\left.-\frac{p}{4\pi i\|\phi\|}\int_{\partial(r\mathbb{D})}(zh)'S_n(F)\overline{zK}\,dz\right].$$

Since F is in H^p and both k and K are in H^q, the above limit equals

$$\frac{1}{2\pi i\|\phi\|}\int_{\partial\mathbb{D}}(zh)'F\overline{zK}\,dz+\frac{ip}{4\pi\|\phi\|}\int_{\partial\mathbb{D}}zhS_n(F)\overline{k}\,d\overline{z}$$
$$-\frac{p}{4\pi i\|\phi\|}\int_{\partial\mathbb{D}}(zh)'S_n(F)\overline{zK}\,dz$$
$$=\frac{1}{2\pi\|\phi\|}\int_0^{2\pi}(zh)'F\overline{K}+S_n(F)\left(\frac{p}{2}h\overline{k}-\frac{p}{2}(zh)'\overline{K}\right)d\theta.$$

We let $n\to\infty$ in the above expression to reach the desired conclusion. □

Taking $h=1$, we have the following corollary, which we call the "norm-equality".

Corollary 3.4. (The Norm-Equality). *Let p be an even integer, let $k\in H^q$, and let F be the extremal function for k. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}|F(e^{i\theta})|^p\,d\theta=\frac{1}{2\pi\|\phi\|}\int_0^{2\pi}F\left[\left(\frac{p}{2}\right)\overline{k}+\left(1-\frac{p}{2}\right)\overline{K}\right]d\theta.$$

The norm-equality is useful mainly because it yields the following theorem.

Theorem 3.5. *Let p be an even integer. Let $\{k_n\}$ be a sequence of H^q functions, and let $k_n\to k$ in H^q. Let F_n be the A^p extremal function for k_n and let F be the A^p extremal function for k. Then $F_n\to F$ in H^p.*

Note that Ryabykh's theorem shows that each $F_n \in H^p$, and that $F \in H^p$. But because the operator taking a kernel to its extremal function is not linear, one cannot apply the closed graph theorem to conclude that $F_n \to F$.

To prove Theorem 3.5 we will use the following lemma involving the notion of uniform convexity. By $x_n \rightharpoonup x$, we mean that x_n approaches x weakly.

Lemma 3.6. *Suppose that X is a uniformly convex Banach space, that $x \in X$, and that $\{x_n\}$ is a sequence of elements of X. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ in X.*

This lemma is known. For example, it is contained in Exercise 15.17 in [14].

Proof of Theorem. We will first show that $F_n \rightharpoonup F$ in H^p (that is, F_n converges to F weakly in H^p). Next we will use this fact and the norm-equality to show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$. By the lemma, it will then follow that $F_n \to F$ in H^p.

To prove that $F_n \rightharpoonup F$ in H^p, note that Ryabykh's theorem says that $\|F_n\|_{H^p} \leq C(\|k_n\|_{H^q}/\|k_n\|_{A^q})^{1/(p-1)}$. Let $\alpha = \inf_n \|k_n\|_{A^q}$ and $\beta = \sup_n \|k_n\|_{H^q}$. Here $\alpha > 0$ because by assumption none of the k_n are identically zero, and they approach k, which is not identically 0. Therefore $\|F_n\|_{H^p} \leq C(\beta/\alpha)^{1/(p-1)}$, and the sequence $\{F_n\}$ is bounded in H^p norm.

Now, suppose that $F_n \not\rightharpoonup F$. Then there is some $\psi \in (H^p)^*$ such that $\psi(F_n) \not\to \psi(F)$. This implies $|\psi(F_{n_j}) - \psi(F)| \geq \epsilon$ for some $\epsilon > 0$ and some subsequence $\{F_{n_j}\}$. But since the sequence $\{F_n\}$ is bounded in H^p norm, the Banach-Alaoglu theorem implies that some subsequence of $\{F_{n_j}\}$, which we will also denote by $\{F_{n_j}\}$, converges weakly in H^p to some function $\widetilde{F}$. Then $|\psi(\widetilde{F}) - \psi(F)| \geq \epsilon$. Now $k_n \to k$ in A^q, and by Theorem 2.6 this implies $F_n \to F$ in A^p, which implies $F_n(z) \to F(z)$ for all $z \in \mathbb{D}$. Since point evaluation is a bounded linear functional on H^p, we have that $F_{n_j}(z) \to \widetilde{F}(z)$ for all $z \in \mathbb{D}$, which means that $\widetilde{F}(z) = F(z)$ for all $z \in \mathbb{D}$. But this contradicts the assumption that $\psi(\widetilde{F}) \neq \psi(F)$. Hence $F_n \rightharpoonup F$.

Let ϕ_n be the functional with kernel k_n, and let ϕ be the functional with kernel k. To show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$, recall that the norm-equality says

$$\frac{1}{2\pi}\int_0^{2\pi} |F_n(e^{i\theta})|^p \, d\theta = \frac{1}{2\pi\|\phi_n\|}\int_0^{2\pi} F_n \left[\left(\frac{p}{2}\right)\overline{k_n} + \left(1 - \frac{p}{2}\right)\overline{K_n}\right] \, d\theta.$$

But, if h is any function analytic in $\mathbb{D}$ and $H(z) = (1/z)\int_0^z h(\zeta)d\zeta$, it can be shown that $\|H\|_{H^q} \leq \|h\|_{H^q}$, as in the proof of Theorem 2.10. Since $k_n \to k$ in H^q, it follows that $K_n \to K$ in H^q. Also, $k_n \to k$ in A^p implies that $\|\phi_n\| \to \|\phi\|$. In addition, $\|F_n\|_{H^p} \leq C$ for some constant C, and $F_n \rightharpoonup F$, so the right-hand side of the above equation approaches

$$\frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F \left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)\overline{K}\right] \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p \, d\theta.$$

In other words, $\|F_n\|_{H^p} \to \|F\|_{H^p}$, and so by Lemma 3.6 we conclude that $F_n \to F$ in H^p. $\square$

3.2 Fourier Coefficients of $|F|^p$

Theorem 3.3 can also be used to gain information about the Fourier coefficients of $|F|^p$, where F is the extremal function. In particular, it leads to a criterion for F to be in L^∞ in terms of the Taylor coefficients of the kernel k.

Theorem 3.7. *Let p be an even integer. Let $k \in H^q$, let F be the A^p extremal function for k, and define K by equation* (3.1)*. Then for any integer $m \geq 0$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p e^{im\theta} \, d\theta = \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F e^{im\theta} \left[\left(\frac{p}{2}\right)\overline{k} + \left(1 - \frac{p}{2}\right)(m+1)\overline{K}\right] \, d\theta.$$

Proof. Take $h(e^{i\theta}) = e^{im\theta}$ in Theorem 3.3. $\square$

This last formula can be applied to obtain estimates on the size of the Fourier coefficients of $|F|^p$.

Theorem 3.8. *Let p be an even integer. Let $k \in A^q$, and let F be the A^p extremal function for k. Let*

$$b_m = \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p e^{-im\theta}\, d\theta,$$

and let

$$k(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then, for each $m \geq 0$,

$$|b_m| = |b_{-m}| \leq \frac{p}{2\|\phi\|} \|F\|_{H^2} \left[\sum_{n=m}^{\infty} |c_n|^2 \right]^{1/2}.$$

Proof. The theorem is trivially true if $k \notin H^2$, so we may assume that $k \in A^2 \subset A^q$. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $F \in H^p$, and $p \geq 2$, we have $F \in H^2$. Now, using Theorem 3.7, we find that

$$\begin{aligned}
b_{-m} &= \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p e^{im\theta}\, d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} (F e^{im\theta}) \left[\left(\frac{p}{2}\right) \overline{k} + \left(1 - \frac{p}{2}\right)(m+1)\overline{K} \right] d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n e^{i(n+m)\theta} \right] \left[\sum_{j=0}^{\infty} \left(\left(\frac{p}{2}\right) \overline{c_j} + \frac{m+1}{j+1} \left(1 - \frac{p}{2}\right) \overline{c_j} \right) e^{-ij\theta} \right] d\theta \\
&= \frac{1}{\|\phi\|} \sum_{n=0}^{\infty} a_n \left(\left(\frac{p}{2}\right) \overline{c_{n+m}} + \frac{m+1}{n+m+1} \left(1 - \frac{p}{2}\right) \overline{c_{n+m}} \right).
\end{aligned}$$

The Cauchy-Schwarz inequality now gives

$$\begin{aligned}
|b_{-m}| &\leq \frac{1}{\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} \left[\sum_{n=m}^{\infty} \left| \left(\frac{p}{2}\right) \overline{c_n} + \frac{m+1}{n+1} \left(1 - \frac{p}{2}\right) \overline{c_n} \right|^2 \right]^{1/2} \\
&\leq \frac{p}{2\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} \left[\sum_{n=m}^{\infty} |c_n|^2 \right]^{1/2}.
\end{aligned}$$

Since

$$\left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} = \|F\|_{H^2}$$

the theorem follows. □

The estimate in Theorem 3.8 can be used to obtain information about the size of $|F|^p$ and F, as in the following corollary.

Corollary 3.9. *If $c_n = O(n^{-\alpha})$ for some $\alpha > 3/2$, then $F \in H^\infty$.*

Proof. First observe that

$$\sum_{n=m}^{\infty} (n^{-\alpha})^2 \le \int_{m-1}^{\infty} x^{-2\alpha}\, dx = \frac{(m-1)^{1-2\alpha}}{2\alpha - 1}.$$

By hypothesis it follows that

$$\left[\sum_{n=m}^{\infty} |c_n|^2\right]^{1/2} = O(m^{(1-2\alpha)/2}).$$

Thus, Theorem 3.8 shows that $b_m = O(m^{(1-2\alpha)/2})$. Therefore $\{b_m\} \in \ell^1$ if $\alpha > 3/2$. But $\{b_m\} \in \ell^1$ implies $|F|^p \in L^\infty$, which implies $F \in H^\infty$. □

In fact, $\{b_m\} \in \ell^1$ implies that $|F|^p$ is continuous in $\overline{\mathbb{D}}$, but this does not necessarily mean F will be continuous in $\overline{\mathbb{D}}$. There is a result similar to Corollary 3.9 in [18], where the authors show that if the kernel k is a polynomial, or even a rational function with no poles in $\overline{\mathbb{D}}$, then F is Hölder continuous in $\overline{\mathbb{D}}$. Their technique relies on deep regularity results for partial differential equations and applies to $1 < p < \infty$. Our result only shows that $F \in H^\infty$, but it applies to a broader class of kernels when p is an even integer.

3.3 Relations Between the Size of the Kernel and Extremal Function

In this section we show that if p is an even integer and $q \le q_1 < \infty$, then the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$. For $q_1 = q$ the statement reduces to Ryabykh's theorem and its previously unknown converse. The following theorem is crucial to the proof.

Theorem 3.10. *Let p be an even integer and let $q = p/(p-1)$ be its conjugate exponent. Let $F \in A^p$ be the extremal function corresponding to the kernel $k \in A^q$. Suppose that $k \in H^{q_1}$ for some q_1 with $q \le q_1 < \infty$, and that $F \in H^{p_1}$, for some p_1 with $p \le p_1 < \infty$. Define p_2 by*

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

If $p_2 < \infty$, then for every trigonometric polynomial h we have

$$\left| \int_0^{2\pi} |F|^p h(e^{i\theta})\, d\theta \right| \le C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \|F\|_{H^{p_1}} \|h\|_{L^{p_2}},$$

where C is some constant depending only on p, p_1, and q_1.

The excluded case $p_2 = \infty$ occurs if and only if $q = q_1$ and $p = p_1$. The theorem is then a trivial consequence of Ryabykh's theorem.

Proof of Theorem. First let h be an analytic trigonometric polynomial. In the proof of Theorem 3.3, we showed that

$$(3.3) \qquad \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta})\, d\theta = \lim_{n\to\infty} \frac{1}{\pi\|\phi\|} \int_{\mathbb{D}} \left((hz)'F + \frac{p}{2} hz(S_nF)' \right) \overline{k}\, dA(z).$$

An application of Lemma 3.2 gives

$$\lim_{n\to\infty} \int_{\mathbb{D}} hz(S_nF)'\overline{k}\, dA = \text{p. v.} \int_{\mathbb{D}} hzF'\overline{k}\, dA,$$

so that the right-hand side of equation (3.3) becomes

$$\frac{1}{\pi\|\phi\|} \text{p. v.} \int_{\mathbb{D}} \left((hz)'F + \frac{p}{2} hzF' \right) \overline{k}\, dA(z).$$

Apply Lemma 3.2 separately to the two parts of the integral to conclude that its absolute value is bounded by

$$C \frac{1}{\|\phi\|} \|k\|_{H^{q_1}} \|f\|_{H^{p_1}} \|h\|_{H^{p_2}},$$

where C is a constant depending only on p_1 and q_1. Since

$$\frac{1}{\|\phi\|} \leq \frac{C_p}{\|k\|_{A^q}}$$

by equation (1.4), this gives the desired result for the special case where h is an analytic trigonometric polynomial.

Now let h be an arbitrary trigonometric polynomial. Then $h = h_1 + \overline{h_2}$, where h_1 and h_2 are analytic polynomials, and $h_2(0) = 0$. Note that the Szegő projection S is bounded from L^{p_2} into H^{p_2} because $1 < p_2 < \infty$. Thus,

$$\|h_1\|_{H^{p_2}} = \|S(h)\|_{H^{p_2}} \leq C\|h\|_{L^{p_2}}.$$

Also,

$$\|h_2\|_{H^{p_2}} = \|zS(e^{-i\theta}\overline{h})\|_{H^{p_2}} = \|S(e^{-i\theta}\overline{h})\|_{H^{p_2}} \leq C\|e^{-i\theta}\overline{h}\|_{L^{p_2}} = C\|h\|_{L^{p_2}},$$

and so

$$\|h_1\|_{H^{p_2}} + \|h_2\|_{H^{p_2}} \leq C\|h\|_{L^{p_2}}.$$

Therefore, by what we have already shown,

$$\begin{aligned}
\left|\int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta})\, d\theta\right| &= \left|\int_0^{2\pi} |F(e^{i\theta})|^p (h_1(e^{i\theta}) + \overline{h_2(e^{i\theta})})\, d\theta\right| \\
&\leq \left|\int_0^{2\pi} |F|^p h_1\, d\theta\right| + \left|\overline{\int_0^{2\pi} |F|^p h_2\, d\theta}\right| \\
&\leq C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|F\|_{H^{p_1}}(\|h_1\|_{H^{p_2}} + \|h_2\|_{H^{p_2}}) \\
&\leq C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|F\|_{H^{p_1}}\|h\|_{L^{p_2}}. \qquad \square
\end{aligned}$$

For a given q_1, we will apply the above theorem with $p_1 = (p-1)q_1$, which then implies that $p_1 = pp_2'$, where p_2' is the conjugate exponent to p_2. This will allow us to bound the H^{p_1} norm of F in terms of $\|\phi\|$ and $\|k\|_{H^{q_1}}$.

Theorem 3.11. *Let p be an even integer, and let q be its conjugate exponent. Let $F \in A^p$ be the extremal function for a kernel $k \in A^q$. If, for q_1 such that $q \le q_1 < \infty$, the kernel $k \in H^{q_1}$, then $F \in H^{p_1}$ for $p_1 = (p-1)q_1$. In fact,*

$$\|F\|_{H^{p_1}} \le C \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)},$$

where C depends only on p and q_1.

Proof. The case $q_1 = q$ is Ryabykh's theorem, so we assume $q_1 > q$. Set $p_1 = (p-1)q_1$. Then $p_1 > p = (p-1)q$. Choose p_2 so that

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

This implies that $p_2 = p_1/(p_1 - p)$, and so its conjugate exponent $p_2' = p_1/p$. Note that $1 < p_2 < \infty$. Let F_n denote the extremal function corresponding to the kernel $S_n k$, which does not vanish identically if n is chosen sufficiently large. Since $S_n k$ is a polynomial, F_n is in H^∞ (and thus $F_n \in H^{p_1}$) by Corollary 3.9. Hence for any trigonometric polynomial h, Theorem 3.10 yields

$$\left| \frac{1}{2\pi} \int_0^{2\pi} |F_n|^p h(e^{i\theta})\, d\theta \right| \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}} \|h\|_{L^{p_2}}.$$

Since the trigonometric polynomials are dense in $L^{p_2}(\partial\mathbb{D})$, taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{p_2}} \le 1$ gives

$$\| |F_n|^p \|_{L^{p_2'}} \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}},$$

which implies

$$\begin{aligned} \|F_n\|_{H^{p_1}}^p &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} (|F_n(e^{i\theta})|^p)^{p_2'}\, d\theta \right\}^{1/p_2'} = \| |F_n|^p \|_{L^{p_2'}} \\ &\le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}}, \end{aligned}$$

since $pp_2' = p_1$. Because $\|F_n\|_{H^{p_1}} < \infty$, we may divide both sides of the inequality by $\|F_n\|_{H^{p_1}}$ to obtain

$$\|F_n\|_{H^{p_1}}^{p-1} \leq C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}},$$

where C depends only on p and q_1. In other words,

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F_n(re^{i\theta})|^{p_1} \, d\theta \right)^{(p-1)/p_1} \leq C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}}$$

for all $r < 1$ and for all n sufficiently large. Note that $S_n k \to k$ in H^{q_1} and in A^q. Since $S_n k \to k$ in A^q, Theorem 2.6 says that $F_n \to F$ in A^p, and thus $F_n \to F$ uniformly on compact subsets of $\mathbb{D}$. Thus, letting $n \to \infty$ in the last inequality gives

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^{p_1} d\theta \right)^{(p-1)/p_1} \leq C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}$$

for all $r < 1$. In other words,

$$\|F\|_{H^{p_1}} \leq \left(C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)}. \qquad \square$$

The remark following Theorem 2.8 implies that a function $F \in A^p$ with unit norm has a corresponding kernel $k \in A^q$ such that F is the extremal function for k, and this kernel is uniquely determined up to a positive multiple. Theorem 3.11 says that if p is an even integer and a kernel k belongs not only to the Bergman space A^q but also to the Hardy space H^{q_1} for some q_1 where $q \leq q_1 < \infty$, then the A^p extremal function F associated with it is actually in H^{p_1} for $p_1 = (p-1)q_1 \geq p$. It is natural to ask whether the converse is true. In other words, if $F \in H^{p_1}$ for some p_1 with $p \leq p_1 < \infty$, must it follow that the corresponding kernel belongs to H^{q_1}? The following theorem says that this is indeed the case.

Theorem 3.12. *Suppose p is an even integer and let q be its conjugate exponent. Let $F \in A^p$ with $\|F\|_{A^p} = 1$, and let k be a kernel such that F is the extremal function* *for k. If $F \in H^{p_1}$ for some p_1 with $p \le p_1 < \infty$, then $k \in H^{q_1}$ for $q_1 = p_1/(p-1)$, and*

$$\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \le C\|F\|_{H^{p_1}}^{p-1},$$

where C is a constant depending only on p and p_1.

Proof. Let h be a polynomial and let ϕ be the functional in $(A^p)^*$ corresponding to k. Then by Theorem A,

$$\begin{aligned}\frac{1}{\|\phi\|}\int_{\mathbb{D}} \overline{k(z)}(zh(z))'\,d\sigma &= \int_{\mathbb{D}} |F(z)|^{p-1}\operatorname{sgn}(\overline{F(z)})(zh(z))'\,d\sigma \\ &= \int_{\mathbb{D}} \overline{F^{p/2}}F^{(p/2)-1}(zh(z))'\,d\sigma.\end{aligned}$$

By hypothesis, $F^{p/2} \in H^{(2p_1)/p}$ and $F^{(p/2)-1} \in H^{2p_1/(p-2)}$. A simple calculation shows that

$$\frac{1}{q_1'} = \frac{q_1 - 1}{q_1} = \frac{p_1 - p + 1}{p_1}$$

and thus

$$\frac{p}{2p_1} + \frac{p-2}{2p_1} + \frac{1}{q_1'} = 1.$$

Now we will apply the first part of Lemma 3.2 with $f_1 = F^{p/2}$ and $f_2 = F^{(p/2)-1}$ and $f_3 = zh$, and with $2p_1/p$ in place of p_1, and $2p_1/(p-2)$ in place of p_2, and q_1' in place of p_3. Note that this is permitted since $1 < 2p_1/p < \infty$, and $1 < q_1' < \infty$, and $1 < 2p_1/(p-2) \le \infty$. (In fact, we even know that $2p_1/(p-2) < \infty$ unless $p = 2$, which is a trivial case since then $F = k/\|k\|_{A^2}$.) With these choices, Lemma 3.2 gives

$$\begin{aligned}\left|\int_{\mathbb{D}} \overline{F^{p/2}}F^{(p/2)-1}(zh(z))'\,d\sigma\right| &\le C\|F^{p/2}\|_{H^{2p_1/p}}\|F^{p/2-1}\|_{H^{2p_1/(p-2)}}\|zh\|_{H^{q_1'}} \\ &= C\|F\|_{H^{p_1}}^{p/2}\|F\|_{H^{p_1}}^{(p-2)/2}\|h\|_{H^{q_1'}} \\ &= C\|F\|_{H^{p_1}}^{p-1}\|h\|_{H_{q_1'}}.\end{aligned}$$

Since

$$\left| \int_{\mathbb{D}} \overline{k(z)}(zh(z))'\, d\sigma \right| \le C\|\phi\| \|F\|_{H^{p_1}}^{p-1} \|h\|_{H_{q_1'}}$$

for all polynomials h, we may define a continuous linear functional ψ on $H^{q_1'}$ such that

$$\psi(h) = \int_{\mathbb{D}} \overline{k(z)}(zh(z))'\, d\sigma$$

for all analytic polynomials h. Then ψ has an associated kernel in H^{q_1}, which we will call $\widetilde{k}$. Thus, for all $h \in H^{q_1'}$, we have

$$\psi(h) = \frac{1}{2\pi} \int_0^{2\pi} \overline{\widetilde{k}(e^{i\theta})} h(e^{i\theta})\, d\theta.$$

But then the Cauchy-Green theorem gives

$$\begin{aligned}
\int_{\mathbb{D}} \overline{k(z)}(zh(z))'\, d\sigma &= \psi(h) \\
&= \frac{1}{2\pi} \int_{\partial\mathbb{D}} \overline{\widetilde{k}(e^{i\theta})} h(e^{i\theta})\, d\theta = \frac{i}{2\pi} \int_{\partial\mathbb{D}} \overline{\widetilde{k}(z)} h(z) z\, d\overline{z} \\
&= \lim_{r\to 1} \frac{i}{2\pi} \int_{\partial(r\mathbb{D})} \overline{\widetilde{k}(z)} h(z) z\, d\overline{z} = \lim_{r\to 1} \frac{1}{\pi} \int_{r\mathbb{D}} \overline{\widetilde{k}(z)} (zh(z))'\, dA \\
&= \int_{\mathbb{D}} \overline{\widetilde{k}(z)} (zh(z))'\, d\sigma,
\end{aligned} \tag{3.4}$$

where h is any analytic polynomial.

Now, for any polynomial $h(z)$, define the polynomial $H(z)$ so that

$$H(z) = \frac{1}{z} \int_0^z h(\zeta)\, d\zeta.$$

Then substituting $H(z)$ for $h(z)$ in equation (3.4), and using the fact that $(zH)' = h$, we have

$$\int_{\mathbb{D}} \overline{\widetilde{k}(z)} h(z)\, d\sigma = \int_{\mathbb{D}} \overline{k(z)} h(z)\, d\sigma$$

for every polynomial h. But since the polynomials are dense in A^p, and k and $\widetilde{k}$ are both in A^q, which is isomorphic to the dual space of A^p, we must have that $k = \widetilde{k}$, and thus $k \in H^{q_1}$.

Now for any polynomial h,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(e^{i\theta})\, d\theta \le C\|\phi\| \|F\|_{H^{p_1}}^{p-1} \|h\|_{H^{q_1'}},$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(e^{i\theta})\, d\theta \le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|h\|_{H^{q_1'}}$$

by inequality (1.4). But if h is any trigonometric polynomial,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(\theta)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} \left[S(h)(e^{i\theta})\right]\, d\theta \\ &\le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|S(h)\|_{H^{q_1'}} \\ &\le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|h\|_{L^{q_1'}}, \end{aligned}$$

where S denotes the Szegő projection. Taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{q_1'}} \le 1$ and dividing both sides of the inequality by $\|k\|_{A^q}$, we arrive at the required bound. □

The main results of this section can be summarized in the following theorem.

Theorem 3.13. *Suppose that p is an even integer with conjugate exponent q. Let $k \in A^q$ and let F be the A^p extremal function associated with k. Let p_1, q_1 be a pair of numbers such that $q \le q_1 < \infty$ and*

$$p_1 = (p-1)q_1.$$

Then $F \in H^{p_1}$ if and only if $k \in H^{q_1}$. More precisely,

$$C_1 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)} \le \|F\|_{H^{p_1}} \le C_2 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)}$$

where C_1 and C_2 are constants that depend only on p and p_1.

Note that if $p_1 = (p-1)q_1$, then $q \le q_1 < \infty$ is equivalent to $p \le p_1 < \infty$.

3.4 Proof of the Lemmas

We now give the proofs of Lemmas 3.1 and 3.2. These proofs are rather technical and require applications of maximal functions and Littlewood-Paley theory.

Definition 3.14. For a function f analytic in the unit disc, the radial maximal function is defined on the unit circle by

$$f^*(e^{i\theta}) = \sup_{0 \le r < 1} |f(re^{i\theta})|.$$

The following is the simplest form of the Hardy-Littlewood maximal theorem (see for instance [5], p. 12).

Theorem C. (Hardy-Littlewood.) *If $f \in H^p$ for $0 < p \le \infty$, then $f^* \in L^p$ and*

$$\|f^*\|_{L^p} \le C\|f\|_{H^p},$$

where C is a constant depending only on p.

Further results of a similar type may be found in [11].

Definition 3.15. For a function f analytic in the unit disc, the Littlewood-Paley function is

$$g(\theta, f) = \left\{ \int_0^1 (1-r)|f'(re^{i\theta})|^2 dr \right\}^{1/2}.$$

A key result of Littlewood-Paley theory is that the Littlewood-Paley function, like the Hardy-Littlewood maximal function, belongs to L^p if and only if $f \in H^p$. Formally, the result may be stated as follows (see [27], Volume 2, Chapter 14, Theorems 3.5 and 3.19).

Theorem D. (Littlewood-Paley.) *For $1 < p < \infty$, there are constants C_p and B_p depending only on p so that*

$$\|g(\cdot, f)\|_{L^p} \le C_p\|f\|_{H^p}$$

for all functions f analytic in $\mathbb{D}$, and

$$\|f\|_{H^p} \leq B_p \|g(\cdot, f)\|_{L^p}$$

for all functions f analytic in $\mathbb{D}$ such that $f(0) = 0$.

We now apply the Littlewood-Paley theorem to obtain the following result, from which Lemmas 3.1 and 3.2 will follow.

Theorem 3.16. *Suppose $1 < p_1, p_2 \leq \infty$, and let p be defined by $1/p = 1/p_1 + 1/p_2$. Suppose furthermore that $1 < p < \infty$. If $f_1 \in H^{p_1}$ and $f_2 \in H^{p_2}$, and h is defined by*

$$h(z) = \int_0^z f_1(\zeta) f_2'(\zeta)\, d\zeta,$$

then $h \in H^p$ and $\|h\|_{H^p} \leq C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$, where C depends only on p_1 and p_2.

Proof. By the definitions of the Littlewood-Paley function and the Hardy-Littlewood maximal function,

$$\begin{aligned} g(\theta, h) &= \left\{ \int_0^1 (1-r)|f_1(re^{i\theta}) f_2'(re^{i\theta})|^2\, dr \right\}^{1/2} \\ &\leq f_1^*(\theta) \left\{ \int_0^1 (1-r)|f_2'(re^{i\theta})|^2\, dr \right\}^{1/2} \\ &= f_1^*(\theta) g(\theta, f_2). \end{aligned}$$

Therefore, since $h(0) = 0$, Theorem D gives

$$\|h\|_{H^p} \leq C\|g(\cdot, h)\|_{L^p} \leq C\|f_1^*\, g(\cdot, f_2)\|_{L^p}.$$

Applying first Hölder's inequality and then Theorem C, we infer that

$$\|h\|_{H^p} \leq C\|f_1^*\|_{L^{p_1}} \|g(\cdot, f_2)\|_{L^{p_2}} \leq C\|f_1\|_{H^{p_1}} \|g(\cdot, f_2)\|_{L^{p_2}}.$$

If $p_2 < \infty$, Theorem D allows us to conclude that

$$\|h\|_{H^p} \leq C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}.$$

This proves the claim under the assumption that $p_2 < \infty$.

If $p_2 = \infty$, then $p_1 < \infty$ by assumption. Integration by parts gives

$$h(z) = f_1(z)f_2(z) - f_1(0)f_2(0) - \int_0^z f_2(\zeta)f_1'(\zeta)\, d\zeta.$$

The H^p norm of the first term is bounded by $\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, by Hölder's inequality. The second term is bounded by $C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$ for some C, since point evaluation is a bounded functional on Hardy spaces. The H^p norm of the last term is bounded by $C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, by what we have already shown, and thus $\|h\|_{H^p} \leq C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$. □

Theorem 3.16 will now be used together with the Cauchy-Green theorem to prove Lemmas 3.2 and 3.1.

Proof of Lemma 3.2. Define

$$I_r = \int_{r\mathbb{D}} \overline{f_1} f_2 f_3'\, dA \qquad \text{and} \qquad H(z) = \int_0^z f_2(\zeta)f_3'(\zeta)\, d\zeta.$$

Then Theorem 3.16 says that $H \in H^q$ and that $\|H\|_{H^q} \leq C\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}}$, where $\frac{1}{q} = \frac{1}{p_2} + \frac{1}{p_3}$. By the Cauchy-Green formula,

$$I_r = \frac{i}{2}\int_{\partial(r\mathbb{D})} \overline{f_1(z)}H(z)\, d\overline{z}.$$

Since $1/p_1 + 1/q = 1$, Hölder's inequality gives

$$|I_r| = \frac{1}{2}\left|\int_{\partial(r\mathbb{D})} \overline{f_1(z)}H(z)\, d\overline{z}\right| \leq \pi M_{p_1}(f_1, r)M_q(H, r).$$

But since $\|H\|_{H^q} \leq C\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}}$, this shows that

$$|I_r| \leq C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}\|f_3\|_{H^{p_3}},$$

which bounds the principal value in question, assuming it exists.

To show that it exists, note that for $0 < s < r$, the Cauchy-Green formula gives

$$\begin{aligned}
2|I_r - I_s| &= \left| \int_{\partial(r\mathbb{D} - s\mathbb{D})} \overline{f_1(z)} H(z)\, d\overline{z} \right| \\
&= \left| \int_0^{2\pi} \left[r\overline{f_1(re^{i\theta})} H(re^{i\theta}) - s\overline{f_1(se^{i\theta})} H(se^{i\theta}) \right] e^{-i\theta}\, d\theta \right| \\
&\le \left| \int_0^{2\pi} \overline{f_1(re^{i\theta})} \left(rH(re^{i\theta}) - sH(se^{i\theta}) \right) e^{-i\theta}\, d\theta \right| \\
&\quad + \left| \int_0^{2\pi} s \left(\overline{f_1(re^{i\theta})} - \overline{f_1(se^{i\theta})} \right) H(se^{i\theta})\, e^{-i\theta}\, d\theta \right|.
\end{aligned}$$

We let $f_r(z) = f(rz)$. Then Hölder's inequality shows that the expression on the right of the above inequality is at most

$$M_{p_1}(f_1, r)\|rH_r - sH_s\|_{H^q} + s\|(f_1)_r - (f_1)_s\|_{H^{p_1}} M_q(H, s).$$

Since $p_1 < \infty$ and $q < \infty$, we know that $(f_1)_r \to f_1$ in H^{p_1} as $r \to 1$, and $H_r \to H$ in H^q as $r \to 1$ (see [5], p. 21). Thus the above quantity approaches 0 as $r, s \to 1$, which shows that the principal value exists.

For the last part of the lemma, what was already shown gives

$$\begin{aligned}
\text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3'\, d\sigma - \int_{\mathbb{D}} \overline{f_1} f_2 (S_n f_3)'\, d\sigma &= \text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 (f_3 - S_n f_3)'\, d\sigma \\
&\le C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3 - S_n(f_3)\|_{H^{p_3}}.
\end{aligned}$$

By assumption $p_3 > 1$. If also $p_3 < \infty$, then the right hand side approaches 0 as $n \to \infty$, which finishes the proof. □

Proof of Lemma 3.1. We know that $f^{p/2} \in H^2$ and $f^{(p/2)-1} \in H^{2p/(p-2)}$. Since h is a polynomial, we have $f^{(p/2)-1}h \in H^{2p/(p-2)}$. Also,

$$\frac{1}{2} + \frac{p-2}{2p} + \frac{1}{p} = 1.$$

Thus, Lemma 3.2 with $f_1 = f^{p/2}$, and $f_2 = f^{(p/2)-1}h$, and $f_3 = f$ gives the result.

□

CHAPTER IV

Explicit Solutions of Some Extremal Problems

4.1 Relation of the Bergman Projection to Extremal Problems

In this chapter we show how information about the Bergman projection can be used to solve certain extremal problems. Recall from Chapter I that the Bergman projection $\mathcal{P}$ is defined by

$$\mathcal{P}(f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^2}\, d\sigma(\zeta), \qquad\qquad f \in L^1(\mathbb{D}),$$

and that $\mathcal{P}$ is the orthogonal projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. We begin with a basic theorem.

Theorem 4.1. *Suppose that* $1 < p < \infty$ *and let* $f \in A^p$ *and* $g \in L^q$*, where* $1/p + 1/q = 1$*. Then*

$$\int_{\mathbb{D}} f\overline{g}\, d\sigma = \int_{\mathbb{D}} f\, \overline{\mathcal{P}(g)}\, d\sigma.$$

Proof. Since $\mathcal{P}$ is the orthogonal projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$, we have for f in A^2 and g in L^2 that

$$\int_{\mathbb{D}} f\overline{g}\, d\sigma = \int_{\mathbb{D}} f\, \overline{[\mathcal{P}(g) + (g - \mathcal{P}(g))]}\, d\sigma = \int_{\mathbb{D}} f\, \overline{\mathcal{P}(g)}\, d\sigma,$$

since $g - \mathcal{P}(g)$ is orthogonal to A^2. Thus, the theorem is true for $p = 2$.

Now, suppose that $p < 2$. Let $\{f_n\}$ be a sequence of A^2 functions approaching f in the A^p norm. Then since $L^q \subset L^2$,

$$\int_{\mathbb{D}} f_n \overline{g}\, d\sigma = \int_{\mathbb{D}} f_n \overline{\mathcal{P}(g)}\, d\sigma,$$

for all n. Since $\mathcal{P}(g) \in L^q$, we may take the limit as $n \to \infty$, which gives

$$\int_{\mathbb{D}} f \overline{g}\, d\sigma = \int_{\mathbb{D}} f \overline{\mathcal{P}(g)}\, d\sigma.$$

If $p > 2$, let $\{g_n\}$ be a sequence of A^2 functions approaching g in the A^q norm. Then since $A^p \subset A^2$,

$$\int_{\mathbb{D}} f \overline{g_n}\, d\sigma = \int_{\mathbb{D}} f \overline{\mathcal{P}(g_n)}\, d\sigma$$

for all n. But $\mathcal{P}(g_n) \to \mathcal{P}(g)$ in the A^q norm because the Bergman projection is continuous from L^q into A^q. Thus,

$$\int_{\mathbb{D}} f \overline{g}\, d\sigma = \int_{\mathbb{D}} f \overline{\mathcal{P}(g)}\, d\sigma. \qquad \square$$

The next theorem gives the first indication of how the Bergman projection is related to extremal problems.

Theorem 4.2. *Suppose that $1 < p < \infty$. Let $F \in A^p$ with $\|F\|_{A^p} = 1$. Then F is the extremal function for the functional with kernel*

$$\mathcal{P}(|F|^{p-1} \operatorname{sgn} F) \in A^q.$$

Furthermore, if F is the extremal function for some functional $\phi \in (A^p)^*$ with kernel $k \in A^q$, then

$$\mathcal{P}(|F|^{p-1} \operatorname{sgn} F) = \frac{k}{\|\phi\|}.$$

Proof. Consider the functional $\psi \in (A^p)^*$ that takes a function $f \in A^p$ to

$$\psi(f) = \int_{\mathbb{D}} f |F|^{p-1} \overline{\operatorname{sgn} F}\, d\sigma.$$

This functional has norm at most $\| \, |F|^{p-1}\overline{\operatorname{sgn} F} \, \|_{L^q} = \|F\|_{L^p}^{p/q} = 1$. But also $\psi(F) = \|F\|_{A^p}^p = 1$, so ψ has norm exactly 1 and F is the extremal function for ψ.

But from Theorem 4.1, it follows that

$$\int_{\mathbb{D}} f \, \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)} \, d\sigma = \int_{\mathbb{D}} f |F|^{p-1} \overline{\operatorname{sgn} F} \, d\sigma$$

for any $f \in A^p$, which means that the kernel in A^q representing ψ is $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$. This proves the first part of the theorem.

If F is also the extremal function for ϕ, it follows that ψ is a positive scalar multiple of ϕ, from the remark following the proof of Theorem 2.8. Since $\|\psi\| = 1$ and ψ is a positive scalar multiple of ϕ, it must be that $\psi = \phi/\|\phi\|$. But this implies that $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F) = k/\|\phi\|$. $\square$

The next two theorems describe the relation of the Bergman projection to a sort of generalized minimal interpolation problem. We will first need the following lemma.

Lemma 4.3. *Let V be a vector space over $\mathbb{C}$, and let $\phi, \phi_1, \dots, \phi_N$ be linear functionals on V such that, for $v \in V$, if $\phi_1(v) = \cdots = \phi_N(v) = 0$, then $\phi(v) = 0$. Then $\phi = \sum_{j=1}^N c_j \phi_j$ for some constants c_j.*

The statement and proof of this lemma may be found in [4] in Appendix A.2 as Proposition 1.4, and in [2] as Lemma 14.

Theorem 4.4. *Let $1 < p < \infty$ and let $\phi_1, \phi_2, \dots, \phi_N \in (A^p)^*$ be linearly independent. Then a function $F \in A^p$ satisfies*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \dots, \phi_N(f) = \phi_N(F)\}$$

if and only if $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \dots, \phi_N$.

Note that this theorem gives a necessary and sufficient condition for a function F to solve the minimal interpolation problem of finding a function $f \in A^p$ of smallest norm such that $\phi_j(f) = c_j$ for $1 \le j \le N$, where $\phi_j \in (A^p)^*$ are arbitrary linearly independent functionals and the c_j are given constants. Namely, F solves the problem if and only if $\phi_j(F) = c_j$ for $1 \le j \le N$ and $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \ldots, \phi_N$. Note that for the case $1 < p < \infty$, the problem under discussion will always have a unique solution, by Proposition 2.3.

Proof. Let $k_1, \ldots, k_N$ be the kernels of $\phi_1, \ldots, \phi_N$, respectively. Suppose that

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}$$

and let h be any non-zero A^p function such that $\phi_1(h) = \cdots = \phi_N(h) = 0$. Since there are only a finite number of the ϕ_j, it is clear that such a function exists. Then $F + h$ is also in contention to solve the extremal problem, so $\|F\| \le \|F + h\|$. Now Theorem B shows that

$$\int_{\mathbb{D}} |F|^{p-1} \overline{\operatorname{sgn} F}\, h\, d\sigma = 0,$$

and so by Theorem 4.1

$$\int_{\mathbb{D}} \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)}\, h\, d\sigma = 0.$$

Define

$$\psi(f) = \int_{\mathbb{D}} \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)}\, f\, d\sigma, \qquad f \in A^p.$$

Lemma 4.3 now shows that

$$\psi = \sum_{j=1}^{N} c_j \phi_j,$$

for some constants c_j, so $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of $k_1, \ldots, k_n$. This proves the "only if" part of the theorem.

Conversely, suppose $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of $k_1, \ldots, k_n$. Then

$$\|F\|_{A^p}^p = \int_{\mathbb{D}} F |F|^{p-1} \overline{\operatorname{sgn} F}\, d\sigma = \int_{\mathbb{D}} F\, \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)}\, d\sigma, \tag{4.1}$$

by Theorem 4.1. Now let $h \in A^p$ be such that $\phi_j(h) = 0$ for $1 \le j \le N$. Since $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the k_j, equation (4.1) gives

$$\begin{aligned}
\|F\|_{A^p}^p &= \int_{\mathbb{D}} (F+h)\overline{\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)}\, d\sigma \\
&= \int_{\mathbb{D}} (F+h)|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma \\
&\le \|F+h\|_{A^p} \||F|^{p-1}\overline{\operatorname{sgn} F}\|_{A^q} \\
&= \|F+h\|_{A^p}\|F\|_{A^p}^{p-1}.
\end{aligned}$$

Therefore,

$$\|F\|_{A^p} \le \|F+h\|_{A^p}.$$

Since h was an arbitrary A^p function with the property that $\phi_j(h) = 0$ for $1 \le j \le N$, this shows that F solves the extremal problem in question. $\square$

When we apply this theorem, we will usually have each ϕ_j be a derivative-evaluation functional. By derivative-evaluation functional, we mean a functional defined by $f \mapsto f^{(n)}(z_0)$ for some integer $n \ge 0$ and some $z_0 \in \mathbb{D}$.

The next theorem is really another form of Theorem 4.4.

Theorem 4.5. *Let $\phi_1, \phi_2, \ldots, \phi_N \in (A^p)^*$ be linearly independent. Let $F \in A^p$. Then the following are equivalent.*

1. F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}$$

but does not satisfy

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_{j_1}(f) = \phi_{j_1}(F), \ldots, \phi_{j_M}(f) = \phi_{j_M}(F)\}$$

for any proper subsequence $\{j_k\}_{k=1}^M$ of $1, 2, \ldots, N$.

2. $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \ldots, \phi_N$, and none of the coefficients in the linear combination is 0.

Proof. Note that if F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_{j_1}(f) = \phi_{j_1}(F), \dots, \phi_{j_M}(f) = \phi_{j_M}(F)\}$$

for some proper subsequence $\{j_k\}_{k=1}^M$ of $1, 2, \dots, N$, then the previous theorem shows that $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of the ϕ_{j_k}.

Conversely, if $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of the functionals ϕ_{j_k}, where $\{j_k\}_{k=1}^M$ is a proper subsequence of $1, 2, \dots, N$, then the previous theorem shows that F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_{j_1}(f) = \phi_{j_1}(F), \dots, \phi_{j_M}(f) = \phi_{j_M}(F)\}. \qquad \square$$

The next two theorems are special cases of Theorems 4.4 and 4.5, with the functionals taken to be $\phi_j(h) = h^{(j)}(0)$, with kernel $k_j(z) = (j+1)!z^j$.

Theorem 4.6. *The function $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a polynomial of degree at most N if and only if*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f(0) = F(0), \dots, f^{(N)}(0) = F^{(N)}(0)\}.$$

Theorem 4.7. *The function $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a polynomial of degree N if and only if N is the smallest integer such that*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f(0) = F(0), \dots, f^{(N)}(0) = F^{(N)}(0)\}.$$

The next theorem relates the generalized minimal interpolation problems we have been discussing with extremal problems.

Theorem 4.8. *Let $\phi_1, \dots, \phi_N$ be linearly independent elements of $(A^p)^*$ with kernels $k_1, \dots, k_N$ respectively, and let $F \in A^p$ with $\|F\|_{A^p} = 1$. Then the functional for which F is the extremal function has as its kernel a linear combination of the k_j if and only if*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \dots, \phi_N(f) = \phi_N(F)\}.$$

This follows from Theorems 4.2 and 4.4. Recall that although there is no unique functional for which F is the extremal function, such a functional is unique up to a positive scalar multiple, which does not affect whether its kernel is a linear combination of the k_j.

One direction of this theorem, the fact that if F is the extremal function for some kernel which is a linear combination of the k_j, then F solves the stated minimal interpolation problem, is easy to prove directly. The proof is as follows. Let F be the extremal function for the functional ϕ, which we assume to have kernel $k = \sum_{j=1}^{N} a_j k_j$. Then

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi(f) = \phi(F)\}.$$

But if some function G in A^p satisfies $\phi_j(F) = \phi_j(G)$ for all j with $1 \leq j \leq N$, then $\phi(G) = \phi(F)$, which implies that $\|F\|_{A^p} \leq \|G\|_{A^p}$. This implies that F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}.$$

We have seen that, in deciding whether a function F solves a certain extremal problem, it may be helpful to consider the Bergman projection $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$. When doing so, it is often convenient to note that $|F|^{p-1} \operatorname{sgn} F = F^{p/2}\overline{F}^{(p/2)-1}$. Here the branch of $F^{p/2}$ does not matter, provided that $F^{(p/2)-1} = F^{p/2}F^{-1}$.

4.2 Calculating Bergman Projections

Now that we have explored the relation between the Bergman projection and solutions to extremal problems, we will calculate the Bergman projection in various cases.

Proposition 4.9. *Let m and n be nonnegative integers. Then*

$$\mathcal{P}(z^m\overline{z}^n) = \begin{cases} \frac{m-n+1}{m+1} z^{m-n}, & \text{if } m \geq n, \\ 0, & \text{if } m < n. \end{cases}$$

This is Lemma 6 in Chapter 2 of [9].

The next theorem is very helpful in calculating the Bergman projection of the kernel of a derivative-evaluation functional times the conjugate of an A^p function.

Theorem 4.10. *Let $1 < q_1, q_2 \leq \infty$. Let p_1 and p_2 be the conjugate exponents of q_1 and q_2. Let*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$$

and suppose that $1 < q < \infty$. Let p be the conjugate exponent of q. Suppose that $k \in A^{q_1}$ and that $g \in A^{q_2}$. Let the functional ψ be defined by $\psi(f) = \int_{\mathbb{D}} f\overline{k}\, d\sigma$ for all $f \in A^{p_1}$. Then $\mathcal{P}(k\overline{g})$ is the kernel of the functional $\phi \in (A^p)^*$ defined by

$$\phi(f) = \psi(fg), \qquad f \in A^p.$$

Proof. First note that $1/p + 1/q_1 + 1/q_2 = 1$, so if $f \in A^p$, then $fg \in A^{p_1}$ and the definition of ϕ makes sense. Now observe that

$$\phi(f) = \psi(fg) = \int_{\mathbb{D}} fg\overline{k}\, d\sigma.$$

By Theorem 4.1, this equals

$$\int_{\mathbb{D}} f\overline{\mathcal{P}\left(\overline{g}k\right)}\, d\sigma.$$

□

Proposition 4.11. *Let $g \in A^p$ for $1 < p < \infty$. Then*

$$\mathcal{P}\left(\frac{1}{(1-\overline{a}z)^2}\,\overline{g(z)}\right) = \frac{\overline{g(a)}}{(1-\overline{a}z)^2}.$$

Proof. Since $1/(1-\overline{a}z)^2$ is the Bergman kernel, it follows from Theorem 4.10 that $\mathcal{P}(\overline{g}/(1-\overline{a}z)^2)$ is the kernel of the functional defined by $f \mapsto f(a)g(a)$. But it is clear that this kernel is precisely

$$\frac{\overline{g(a)}}{(1-\overline{a}z)^2}.$$

□

We will study the kernels of various derivative-evaluation functionals. Evaluation at the origin is somewhat exceptional and simpler than evaluation elsewhere, so we deal with it first.

Theorem 4.12. *The kernel for the functional $f \mapsto f^{(n)}(0)$ is $(n+1)!z^n$. If $g \in A^p$ then*

$$\mathcal{P}(z^n\overline{g(z)}) = \sum_{j=0}^{n} \frac{\overline{g^{n-j}(0)}}{(n-j)!}\frac{j+1}{n+1}z^n.$$

Proof. The first statement can be verified by evaluating

$$\int_{\mathbb{D}} f(z)\overline{z}^n d\sigma(z)$$

by writing f as a power series. The second part follows from Proposition 4.9. To see this, note that by the first part of the theorem, $\mathcal{P}(z^n\overline{g(z)})$ is the kernel for the functional taking $f \in A^p$ to

$$\frac{1}{(n+1)!}(fg)^{(n)}(0) = \frac{1}{(n+1)!}\sum_{j=0}^{n}\binom{n}{j} f^{(j)}(0)g^{(n-j)}(0),$$

which has kernel

$$\sum_{j=0}^{n} \frac{\overline{g^{n-j}(0)}}{(n-j)!}\frac{j+1}{n+1}z^j.$$

□

We now study the functional with kernel $1/(1-\overline{a}z)^3$.

Proposition 4.13. *Suppose $a \in \mathbb{D}$ and $a \neq 0$, and let $1 \leq p < \infty$. Then the function*

$$\frac{1}{(1-\overline{a}z)^3}$$

is the kernel for the functional in $(A^p)^*$ *defined by*

$$f \mapsto \frac{a}{2} f'(a) + f(a)$$

Proof. For every $a \in \mathbb{D}$, we have that

$$f(a) = \int_{\mathbb{D}} \frac{1}{(1 - a\overline{z})^2} f(z) \, d\sigma(z).$$

Differentiation with respect to a yields

$$f'(a) = \int_{\mathbb{D}} \frac{2\overline{z}}{(1 - a\overline{z})^3} f(z) \, d\sigma(z),$$

so the kernel for the functional $f \mapsto f'(a)$ is $2z/(1 - \overline{a}z)^3$. But

$$\frac{1}{(1 - \overline{a}z)^3} = \frac{\overline{a}}{2} \frac{2z}{(1 - \overline{a}z)^3} + \frac{1}{(1 - \overline{a}z)^2},$$

so $1/(1 - \overline{a}z)^3$ is the kernel for $f \mapsto (a/2)f'(a) + f(a)$. $\square$

Proposition 4.14. *Let* $1 < p < \infty$, *and let* $g \in A^p$. *Suppose* $a \in \mathbb{D}$ *and* $a \neq 0$. *If* $g(a) = 0$, *then*

$$\mathcal{P}\left(\frac{1}{(1 - \overline{a}z)^3} \, \overline{g} \right) = \frac{\overline{a}}{2} \, \overline{g'(a)} \, \frac{1}{(1 - \overline{a}z)^2}.$$

Proof. By Theorem 4.10,

$$\mathcal{P}\left(\frac{1}{(1 - \overline{a}z)^3} \, \overline{g} \right)$$

is the kernel associated with the functional $f \mapsto \frac{a}{2}(f'(a)g(a) + f(a)g'(a)) + f(a)g(a)$. Since $g(a) = 0$, this equals $(a/2)g'(a)f(a)$. But the kernel for this functional is

$$\frac{\overline{a}}{2} \, \overline{g'(a)} \, \frac{1}{(1 - \overline{a}z)^2}. \qquad \square$$

We will now deal with the function $1/(1 - \overline{a}z)^n$, for $n \geq 2$, which includes the earlier results of this section.

Proposition 4.15. *The kernel for the functional $f \mapsto f^{(n)}(a)$ is*

$$\frac{(n+1)!z^n}{(1-\overline{a}z)^{n+2}}, \qquad n = 0, 1, 2, \ldots$$

Proof. We know that

$$f(a) = \int_{\mathbb{D}} \frac{1}{(1-a\overline{z})^2} f(z)\, d\sigma(z).$$

Differentiation n times with respect to a gives the result. □

Proposition 4.16. *For each $a \in \mathbb{D}$ with $a \neq 0$, there are numbers $c_0, \ldots, c_n$ with $c_n \neq 0$ such that the function*

$$\frac{1}{(1-\overline{a}z)^{n+2}}$$

is the kernel for the functional $f \mapsto c_0 f(a) + c_1 f'(a) + \ldots + c_n f^{(n)}(a)$.

Proof. We will proceed by induction. The claim is true for $n = 0$ by the reproducing property of the Bergman kernel function. For general n, we may write the partial fraction expansion

$$\frac{z^n}{(1-\overline{a}z)^{n+2}} = \sum_{j=0}^{n+2} \frac{b_j}{(1-\overline{a}z)^j},$$

for some complex numbers b_j. Thus,

$$z^n = \sum_{j=0}^{n+2} b_j (1-\overline{a}z)^{n+2-j}.$$

Differentiating both sides $n+1$ times with respect to z gives

$$0 = b_1(-\overline{a})^{n+1}(n+1)! + b_0(-\overline{a})^{n+1}(n+2)!(1-\overline{a}z).$$

Since this holds for all z, it follows that $b_0 = b_1 = 0$. Since $z^n/(1-\overline{a}z)^{n+2}$ has a pole of order $n+2$ at $1/\overline{a}$, we see that $b_{n+2} \neq 0$. Therefore,

$$\frac{1}{(1-\overline{a}z)^{n+2}} = \frac{1}{b_{n+2}} \left(\frac{z^n}{(1-\overline{a}z)^{n+2}} - \sum_{j=2}^{n+1} \frac{b_j}{(1-\overline{a}z)^j} \right).$$

But the first term of the right side of the above equation is a the kernel for the functional $f \mapsto (1/(n+1)!)f^{(n)}(a)$. Also, each term in the sum $\sum_{j=2}^{n+1} \frac{b_j}{(1-\overline{a}z)^j}$ is the kernel for a linear functional taking each function f to some linear combination of $f(a), f'(a), \ldots,$ and $f^{(n-1)}(a)$, by the induction hypothesis. This proves the proposition. $\square$

Proposition 4.17. *Let $g \in A^p$ for $1 < p < \infty$, and let $a \in \mathbb{D}$ with $a \neq 0$. Suppose g has a zero of order n at a. Let $N \geq 0$ be an integer. Then*

$$\mathcal{P}\left(\frac{1}{(1-\overline{a}z)^{N+2}}\overline{g(z)}\right) = \sum_{k=0}^{N-n} C_k \frac{1}{(1-\overline{a}z)^{k+2}}$$

for some complex constants C_k depending on $g^{(m)}(a)$ for $0 \leq m \leq N$.

Proof. The projection

$$\mathcal{P}\left(\frac{1}{(1-\overline{a}z)^{N+2}}\,\overline{g(z)}\right)$$

is the kernel associated with the functional

$$f \mapsto \sum_{j=0}^{N} b_j (fg)^{(j)}(a)$$

for some constants b_j, with $b_N \neq 0$, by the previous proposition and Theorem 4.10. But

$$\sum_{j=0}^{N} b_j (fg)^{(j)}(a) = \sum_{j=0}^{N}\sum_{k=0}^{j} b_j \binom{j}{k} f^{(k)}(a) g^{(j-k)}(a).$$

Since $g^{(j)}(a) = 0$ for $0 \leq j < n$, all terms in the sum with $j - k < n$ are 0. But this means that the only non-zero terms in the sum occur when $k \leq j - n$, so that $k \leq N - n$. Now, set

$$B_k = \sum_{j=k+n}^{N} b_j \binom{j}{k} g^{(j-k)}(a),$$

so

$$\sum_{j=0}^{N} b_j (fg)^{(j)}(a) = \sum_{k=0}^{N-n} B_k f^{(k)}(a).$$

But the kernel associated to $\sum_{k=0}^{N-n} B_k f^{(k)}(a)$ is

$$\sum_{k=0}^{N-n} B_k \frac{(k+1)!z^k}{(1-\overline{a}z)^{k+2}}.$$

But as in the proof of Theorem 4.16, we may show that

$$\frac{z^k}{(1-\overline{a}z)^{k+2}} = \frac{c_{k2}}{(1-\overline{a}z)^2} + \frac{c_{k3}}{(1-\overline{a}z)^3} + \cdots + \frac{c_{k,k+2}}{(1-\overline{a}z)^{k+2}}$$

for some constants $c_{k2}, \ldots, c_{k,k+2}$. Thus we may write

$$\sum_{k=0}^{N-n} B_k \frac{(k+1)!z^k}{(1-\overline{a}z)^{k+2}} = \sum_{k=0}^{N-n} C_k \frac{1}{(1-\overline{a}z)^{k+2}}$$

for some constants C_k, depending on $g^{(m)}(a)$ for $0 \leq m \leq N$. $\square$

We will now deal with the function $1/(1-\overline{a}z)$. Since the functional with kernel $1/(1-\overline{a}z)^{n+2}$ involves differentiation of order n, it seems reasonable that the functional with kernel $1/(1-\overline{a}z)$ involves integration. This is indeed the case.

Proposition 4.18. *The function*

$$1/(1-\overline{a}z)$$

is the kernel for the functional defined on A^p for $1 < p < \infty$ by

$$f \mapsto \frac{1}{a}\int_0^a f(z)\,dz.$$

Proof. Since

$$\frac{1}{1-\overline{a}z} = \sum_{n=0}^{\infty} (\overline{a}z)^n,$$

it follows that

$$\int_{\mathbb{D}} \frac{z^m}{1-a\overline{z}}\,d\sigma = \sum_{n=0}^{\infty} \int_{\mathbb{D}} (a\overline{z})^n z^m\,d\sigma = a^m \int_{\mathbb{D}} |z|^{2m}\,d\sigma = \frac{a^m}{m+1}. \tag{4.2}$$

The change in the order of integration and summation is justified by the fact that the sum converges uniformly in $\overline{\mathbb{D}}$. Now let $f \in A^p$ and write $f(z) = \sum_{m=0}^{\infty} b_m z^m$. Define

$$F(z) = \frac{1}{z}\int_0^z f(\zeta)\,d\zeta = \sum_{m=0}^{\infty} \frac{b_m}{m+1} z^m.$$

Therefore, by equation (4.2),

$$\int_{\mathbb{D}} \frac{1}{1-a\overline{z}} f(z)\,d\sigma = \int_{\mathbb{D}} \frac{1}{1-a\overline{z}} \left(\sum_{m=0}^{\infty} b_m z^m\right) d\sigma = \sum_{m=0}^{\infty} b_m \frac{a^m}{m+1} = F(a).$$

The interchange of the order of integration and summation is justified by the fact the partial sums of the Taylor series for f approach f in A^p. $\square$

Due to their relation with extremal problems, we are often concerned with projections of the form $\mathcal{P}(F^{p/2}\overline{F}^{(p/2)-1})$, where F is an analytic function. This is well defined because

$$F^{p/2}\overline{F}^{(p/2)-1} = |F|^p/\overline{F} = |F|^{p-1}\operatorname{sgn} F.$$

The following theorems deal with this.

Theorem 4.19. *For $1 \le n \le N$, let d_n be a nonnegative integer and let $z_n \in \mathbb{D}$. Let k be analytic and a linear combination of the kernels of the functionals given by $f \mapsto f^{(d_n)}(z_n)$. Let $g \in A^p$ for $p > 1$. Then $\mathcal{P}(k\overline{g})$ is in the linear span of the set of all the kernels of functionals defined by $f \mapsto f^{(m)}(z_n)$, where m is an integer with $0 \le m \le d_n$ and n is an integer with $1 \le n \le N$.*

Proof. Let $k = \sum_{n=1}^{N} a_n k_n$, where k_n is the kernel for the functional $f \mapsto f^{(d_n)}(z_n)$. Then by Theorem 4.10, $\mathcal{P}(k_n\overline{g})$ is the kernel of the functional

$$f \mapsto (fg)^{(d_n)}(z_n) = \sum_{j=0}^{d_n} \binom{d_n}{j} f^{(j)}(z_n) g^{(n-j)}(z_n).$$

But this functional is a linear combination of the functionals of the form

$$f \mapsto f^{(m)}(z_n),$$

where $0 \le m \le d_n$. $\square$

The next theorem shows that if $F^{p/2}$ is an analytic function of a certain form, then F must solve a certain extremal problem.

Theorem 4.20. *Let $1 < p < \infty$. Given a positive integer N, suppose that $F \in A^p$ and $F^{p/2}$ is analytic and a linear combination of the kernels k_n corresponding to the functionals $f \mapsto f^{(d_n)}(z_n)$ for some integers d_n and some points $z_n \in \mathbb{D}$, where $1 \le n \le N$. Then F satisfies*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f^{(m)}(z_n) = F^{(m)}(z_n) \text{ for all } n \text{ and } m \text{ such that } 1 \le n \le N \text{ and } 1 \le m \le d_n\}.$$

This follows from Theorems 4.19 and 4.5.

The next theorem is a consequence of Theorem 4.19. We give an alternate proof, based on a computation involving Proposition 4.9.

Theorem 4.21. *Let f be a polynomial of degree at most N and let $g \in A^p$ for some $p > 1$. Then $\mathcal{P}(f\overline{g})$ is a polynomial of degree at most N.*

Proof. Let $f = \sum_{j=0}^{N} a_j z^j$. Assume first that g is a polynomial and let the integer m be the maximum of N and the degree of g. Write $g = \sum_{k=0}^{m} b_k z^k$, where $b_k = 0$ if $k > \deg(m)$. Then

$$\begin{aligned}
\mathcal{P}\left(\sum_{j=0}^{N} a_j z^j \overline{\sum_{k=0}^{m} b_k z^k}\right) &= \mathcal{P}\left(\sum_{j=0}^{N} a_j z^j \overline{\sum_{k=0}^{N} b_k z^k}\right) + \mathcal{P}\left(\sum_{j=0}^{N} a_j z^j \overline{\sum_{k=N+1}^{m} b_k z^k}\right) \\
&= \sum_{j,k=0}^{N} a_j \overline{b_k} \mathcal{P}\left(z^j \overline{z}^k\right) + \sum_{j=0}^{N} \sum_{k=N+1}^{m} a_j \overline{b_k} \mathcal{P}\left(z^j \overline{z}^k\right) \\
&= \sum_{0 \le k \le j \le N} a_j \overline{b_k} \, \frac{j-k+1}{j+1} \, z^{j-k} + \sum_{0 \le j < k \le N} 0 + \sum_{j=0}^{N} \sum_{k=N+1}^{m} 0.
\end{aligned}$$

The last step follows from Proposition 4.9. Note that the sums from $N+1$ to m are taken to be 0 if $N = m$. This proves the theorem if g is a polynomial.

If g is not a polynomial, let $\{g_n\}$ be a sequence of polynomials that approach g in the A^p norm. Then $\mathcal{P}(f\overline{g_n}) \to \mathcal{P}(f\overline{g})$ in A^p. Since each $\mathcal{P}(f\overline{g_n})$ is a polynomial of degree at most N, so is $\mathcal{P}(f\overline{g})$. □

Using this theorem and Theorem 4.7, we immediately get the following result.

Theorem 4.22. *Suppose that $F \in A^p$ and $F^{p/2}$ is a polynomial of degree N. Then*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f(0) = F(0), \ldots, f^{(N)}(0) = F^{(N)}(0)\}.$$

Note that $F^{p/2}$ can be a polynomial only if $p/2$ is rational and all the zeros of F in $\mathbb{D}$ are of order a multiple of s, where r/s is the reduced form of $p/2$. If p is an even integer, this poses no restriction. Because of this, the case where p is an even integer is often easier to work with.

4.3 Solution of Specific Extremal Problems

We will now discuss how to solve some specific minimal interpolation problems. Since we are dealing with the powers $p/2$ and $2/p$, neither of which need be an integer, we will have to take care in our calculations. We will introduce a lemma to facilitate this. The lemma basically says that if f and g are analytic functions nonzero at the origin, and if $f^{(n)}(0) = (g^p)^{(n)}(0)$ for all n such that $0 \le n \le N$, then $(f^{1/p})^{(n)}(0) = g^{(n)}(0)$ for all n such that $0 \le n \le N$.

To state the lemma we first need to introduce some notation. Suppose that the constants $c_0, c_1, \ldots, c_N$ are given and that $c_0 \neq 0$, and let $h(z) = c_0 + c_1 z + \cdots + c_N z^N$. Suppose that $a = z^p$ for some branch of the function z^p. Let U be a neighborhood of the origin such that $h(U)$ is contained in some half plane whose boundary contains the origin, and such that $0 \notin h(U)$. Then we can define z^p so that it is analytic in $h(U)$ and so that $c_0^p = a$. We let $\beta_j^p(a; c_0, c_1, \ldots, c_j)$ denote the j^{th} derivative of $h(z)^p$

at 0.

Note that because of the chain rule for differentiation, β_j only depends on j, the constants $c_0, \ldots, c_j$, and the numbers p and a. For the same reason, the value of β is the same if we replace the function h in the definition of β by any function $\widetilde{h}$ analytic near the origin such that $\widetilde{h}^{(j)}(0) = c_j$ for $1 \leq j \leq N$.

Lemma 4.23. *Let $c_0, c_1, \ldots, c_N$ be given complex numbers, and let p be a real number. Suppose that $c_0 \neq 0$, and let $a_0 = c_0^p$, for some branch of z^p. Then*

$$c_j = \beta_j^{1/p}\left(c_0; \beta_0^p(a_0; c_0), \beta_1^p(a_0; c_0, c_1), \ldots, \beta_N^p(a_0; c_0, \ldots, c_N)\right).$$

Proof. For ease of notation let $a_j = \beta_j^p(a_0; c_0, \ldots, c_j)$ and $b_j = \beta_j^{1/p}(c_0; a_0, \ldots, a_N)$. Then $b_0 = c_0$.

Now let $f(z) = \sum_{j=0}^{N} \frac{c_j}{j!} z^j$. Then $f^{(j)}(0) = c_j$ for $0 \leq j \leq N$. Let U be a neighborhood of 0 such that there exist $r_0 > 0$ and $\theta_0 \in \mathbb{R}$ such that

$$f(U) \subset \left\{ re^{i\theta} : r_0 < r \text{ and } \theta_0 - \frac{\pi}{2p} < \theta < \theta_0 + \frac{\pi}{2p} \right\}.$$

Then z^p can be defined as an analytic function in $f(U)$. Furthermore, the set $V = (f(U))^p$ does not contain 0 but is contained in some half plane, so $z^{1/p}$ can be defined as an analytic function in V so that it is the inverse of the function z^p defined in $f(U)$.

Now define $g(z) = (f(z))^p$ for $z \in U$. Then $g^{(j)}(0) = a_j$ and $g^{1/p}(0) = c_0$, so $(g^{1/p})^{(j)}(0) = b_j$ for $0 \leq j \leq N$. But $g(z)^{1/p} = f(z)$ for $z \in U$, so $b_j = c_j$ for $0 \leq j \leq N$.

$\square$

We will now use the lemma to solve a specific extremal problem in certain cases.

Theorem 4.24. *Let $c_0, \dots, c_N$ be given complex numbers, and assume that $c_0 \neq 0$. Suppose that $F \in A^p$, and $F^{(j)}(0) = c_j$ for $0 \leq j \leq N$, and*

$$\|F\|_{A^p} = \inf\{\|g\|_{A^p} : g(0) = c_0, \dots, g^{(N)}(0) = c_N\}.$$

Let $a_0 = c_0^{p/2}$ for some branch of z^p. Define

$$f(z) = \sum_{j=0}^{N} \frac{\beta_j^{p/2}(a_0; c_0, \dots, c_j)}{j!} z^j.$$

Suppose that f has no zeros in $\mathbb{D}$. Then we may define $f^{2/p}$ so that it is analytic in $\mathbb{D}$ and so that $f^{2/p}(0) = c_0$. Then

$$F = f^{2/p}.$$

The same result also holds if p is rational, $2/p = r/s$ in lowest form, and every zero of f has order a multiple of s.

Proof. Note that $f^{2/p}$ is analytic in $\mathbb{D}$ since we have assumed f has no zeros in $\mathbb{D}$, or that p is rational and $2/p = r/s$ in lowest form and f has only zeros whose orders are multiples of s. Also, $f(0) = a_0$, so we may define $f^{2/p}$ so that $f^{2/p}(0) = c_0$. The j^{th} derivative of $f^{2/p}$ at 0 is

$$\beta_j^{2/p}\left(c_0; \beta_0^{p/2}(a_0; c_0), \dots, \beta_j^{p/2}(a_0; c_0, \dots, c_j)\right)$$

for $0 \leq j \leq N$, which equals c_j by the lemma. Thus $f^{2/p}$ is in contention to solve the extremal problem.

But

$$\mathcal{P}\left(\frac{|f^{2/p}|^p}{\overline{f}^{2/p}}\right) = \mathcal{P}(f\overline{f}^{1-(2/p)})$$

is a polynomial of degree at most N by Theorem 4.21, so by Theorem 4.22 we find $F = f^{2/p}$. $\square$

To apply this theorem, we need to show that f has no zeros in the unit disc, or has only zeros of suitable orders if p is rational. Then $f^{2/p}$ is the extremal function. Note that we do not need to know anything about the zeros of the extremal function itself to apply the theorem. However, if f has no zeros in the unit disc, this theorem implies that the extremal function $F = f^{2/p}$ also has no zeros in the unit disc. It seems likely that if F has no zeros, then F must equal $f^{2/p}$, but we do not know a proof of this.

Example 4.25. The solution to the minimal interpolation problem in A^p with $f(0) = 1$ and $f'(0) = c_1$ is

$$F(z) = \left(1 + \frac{p}{2}c_1 z\right)^{2/p},$$

provided that $|c_1| \leq \frac{2}{p}$. This is because $\beta_0^{p/2}(1; 1, c_1) = 1$ and $\beta_1^{p/2}(1; 1, c_1) = (p/2)c_1$. For example, if $p = 4$ and $c_1 = \frac{1}{2}$, then

$$F(z) = (1 + z)^{1/2}.$$

The above problem is also solved in [21] in more general form. However, the solution to the extremal problem in the next example was previously unknown.

Example 4.26. The solution to the minimal interpolation problem in A^p for $1 < p < \infty$ with $F(0) = 1$, and $F'(0) = c_1$, and $F''(0) = c_2$ is

$$F(z) = \left\{1 + (p/2)c_1 z + [(p(p-2)/4)c_1^2 + (p/2)c_2]z^2\right\}^{2/p},$$

provided that the quadratic polynomial under the $2/p$ exponent in the equation for F has no zeros in $\mathbb{D}$.

Linear extremal problems tend to be more difficult to solve than minimal interpolation problems involving derivative-evaluation functionals, because values of a function f and its derivatives are generally easier to calculate than $\mathcal{P}(|f|^{p-1} \operatorname{sgn} f)$.

Nevertheless it is possible to solve some linear extremal problems explicitly by the methods in this chapter. Here is one example.

Theorem 4.27. *Let $N \geq 1$ be an integer, let $1 < p < \infty$, and let $b \in \mathbb{C}$ satisfy*

$$|b| \geq 1 + \frac{1}{N+1}\left(1 - \frac{2}{p}\right),$$

and define

$$a = \frac{|b| + \sqrt{|b|^2 - \frac{4}{N+1}\left(1 - \frac{2}{p}\right)}}{2} \operatorname{sgn} b.$$

Then the solution to the extremal problem in A^p with kernel $z^N + b$ is

$$F(z) = \operatorname{sgn}(a^{1-(2/p)}) \frac{(z^N + a)^{2/p}}{(|a|^2 + 1/(N+1))^{1/p}}.$$

In the above expression for $F(z)$, the branch of $(z^N + a)^{2/p}$ may be chosen arbitrarily, but the value of $\operatorname{sgn}(a^{1-(2/p)})$ must be chosen consistently with this choice. Note that the functional associated with the kernel $z^N + b$ is

$$\phi(f) = bf(0) + (1/(N+1)!)f^{(N)}(0).$$

Also, observe that the hypothesis of the theorem holds for all N and p if $|b| \geq 3/2$.

Proof. The condition

$$|b| \geq 1 + \frac{1}{N+1}\left(1 - \frac{2}{p}\right)$$

implies that $|a| \geq 1$, so that $z^N + a \neq 0$ in $\mathbb{D}$ and F is an analytic function. Note that

$$\|(z^N + a)^{2/p}\|_{A^p}^p = \int_{\mathbb{D}} |z^N + a|^2 \, d\sigma = \int_{\mathbb{D}} (z^N + a)\overline{(z^N + a)} \, d\sigma = |a|^2 + \frac{1}{N+1}.$$

Thus, $\|F\|_{A^p} = 1$.

Now

$$((z^N + a)^{2/p})^{p/2-1} = a^{1-2/p} + \left(1 - \frac{2}{p}\right) a^{-2/p} z^N + O(z^{2N}),$$

where we choose branches so that $((z^N+a)^{2/p})^{p/2} = z^N + a$. We calculate that

$$\begin{aligned}
&\mathcal{P}\left(|z^N+a|^{p-1}\operatorname{sgn}(z^N+a)\right)\\
&= \mathcal{P}\left((z^N+a)\overline{(z^N+a)}^{1-2/p}\right)\\
&= \mathcal{P}\left((z^N+a)\left(\overline{a^{1-2/p}} + \left(1-\frac{2}{p}\right)\overline{a^{-2/p}}\,\overline{z^N} + O(\overline{z^{2N}})\right)\right).
\end{aligned}$$

But by Proposition 4.9, this equals

$$\begin{aligned}
&\mathcal{P}\left[(z^N+a)\left(\overline{a^{1-2/p}} + \left(1-\frac{2}{p}\right)\overline{a^{-2/p}}\,\overline{z^N}\right)\right]\\
&= a\overline{a^{1-2/p}} + \frac{1}{N+1}\left(1-\frac{2}{p}\right)\overline{a^{-2/p}} + \overline{a^{1-(2/p)}}z^N\\
&= \overline{a^{1-(2/p)}}\left(z^N + a + \frac{1}{N+1}\left(1-\frac{2}{p}\right)\overline{a}^{\,-1}\right)\\
&= \overline{a^{1-(2/p)}}(z^N+b).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{P}\left\{\left|\operatorname{sgn}(a^{1-(2/p)})(z^N+a)\right|^{p-1}\operatorname{sgn}\left(\operatorname{sgn}(a^{1-(2/p)})(z^N+a)\right)\right\}\\
&= \overline{a^{1-(2/p)}}\operatorname{sgn}(a^{1-(2/p)})(z^N+b)\\
&= |a^{1-(2/p)}|\,(z^N+b).
\end{aligned}$$

Therefore,

$$\mathcal{P}(F^{p/2}\overline{F^{(p/2)-1}}) = |a^{1-(2/p)}|\frac{z^N+b}{(|a|^2+1/(N+1))^{(p-1)/p}}.$$

Since $\|F\|_{A^p} = 1$, Theorem 4.2 shows that F is the extremal function for the kernel on the right of the above equation. But that kernel is a positive scalar multiple of k, so F is also the extremal function for k. $\square$

4.4 Canonical Divisors

We will now discuss how our previous results apply to canonical divisors. Recall that these divisors are the Bergman space analogues of Blaschke products. By the zero-set of an A^p function not identically 0, we mean its collection of zeros, repeated according to multiplicity. Such a set will be countable, since the zeros of analytic functions are discrete. Given an A^p zero set, we can consider the space N^p of all functions that vanish on that set. More precisely, $f \in A^p$ is in N^p if it vanishes at every point in the given zero set, to at least the prescribed multiplicity.

If the zero set does not include 0, we pose the extremal problem of finding $G \in N^p$ such that $\|G\|_{A^p} = 1$, and such that $G(0)$ is positive and as large as possible. If the zero set has a zero of order n at 0, we instead maximize $G^{(n)}(0)$. For $0 < p < \infty$, this problem has a unique solution, which is called the canonical divisor. For $1 < p < \infty$, this follows from the fact that an equivalent problem is to find an $F \in N^p$ with $F(0) = 1$ and $\|F\|_{A^p}$ as small as possible. By Proposition 2.3, the latter problem has a unique solution.

In this section, we discuss the problem of determining the canonical divisor when p is an even integer, and the zero set is finite. We show how the methods of this chapter can be used to characterize the canonical divisor. Our methods show that if G is the canonical divisor, then $G^{p/2}$ is a rational function with residue 0 at each of its poles, which is the content of the following theorem.

Theorem 4.28. *Let p be an even integer. Let $z_1, \ldots, z_N$ be distinct points in $\mathbb{D}$, and consider the zero-set consisting of each of these points with multiplicities $d_1, \ldots, d_N$, respectively. Let G be the canonical divisor for this zero set. Then there are constants c_0 and c_{nj} for $1 \le n \le N$ and $0 \le j \le (p/2)d_n - 1$, such that*

$$G(z)^{p/2} = c_0 + \sum_{n=1}^{N} \sum_{j=0}^{(p/2)d_n - 1} \frac{c_{nj}}{(1 - \overline{z_n} z)^{j+2}}, \qquad \textit{if } z_n \neq 0 \textit{ for all } n, \textit{ and}$$

$$G(z)^{p/2} = c_0 z^{(p/2)d_1} + \sum_{j=0}^{(p/2)d_1 - 1} c_{1j} z^j + \sum_{n=2}^{N} \sum_{j=0}^{(p/2)d_n - 1} \frac{c_{nj}}{(1 - \overline{z_n} z)^{j+2}}, \qquad \textit{if } z_1 = 0.$$

Proof. We will begin by illustrating the argument in the relatively simple case where $p = 4$, each $z_n \neq 0$, and all multiplicities $d_n = 1$. Our goal is to show that $G^{p/2}$ is the kernel for some linear combination of certain derivative-evaluation functionals. Because we know what the kernel of any derivative-evaluation functional is, we will be able to show that G has the form stated in the theorem.

Let

$$h_n(z) = \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{z - z_j}{z_n - z_j}$$

so that $h_n(z_n) = 1$ and $h_n(z_j) = 0$ for $j \neq n$. Let $f \in A^p$. Define

$$\hat{f}(z) = f(z) - \sum_{n=1}^{N} f(z_n) h_n(z).$$

Then $\hat{f}(z_n) = 0$ for $1 \leq n \leq N$. In [9] (as well as [6] and [24]), it is shown that the canonical divisor of a finite zero set has no zeros on $\partial\mathbb{D}$. Also, the canonical divisor of any zero set has no excess zeros; i.e., it has only the prescribed zeros and no more. Thus the function

$$\widetilde{f}(z) = \frac{1}{G(z)} \hat{f}(z)$$

is in A^p. But then

$$\begin{aligned}
\int_{\mathbb{D}} \overline{G^2} f \, d\sigma &= \int_{\mathbb{D}} \overline{G(z)^2} \left(\hat{f}(z) + \sum_{n=1}^{N} f(z_n) h_n(z) \right) d\sigma \\
&= \int_{\mathbb{D}} |G(z)|^3 \, \overline{\operatorname{sgn}(G(z))} \widetilde{f}(z) \, d\sigma + \int_{\mathbb{D}} \overline{G(z)^2} \left(\sum_{n=1}^{N} f(z_n) h_n(z) \right) d\sigma \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

We first deal with the term I. The canonical divisor G is a constant multiple of the function F such that $F(z_n) = 0$ for $1 \leq n \leq N$ and $F(0) = 1$, and such that F has the smallest norm possible. Thus by Theorem 4.4, $\mathcal{P}(|G|^3 \operatorname{sgn}(G))$ is the kernel for a linear combination of point evaluation functionals at the points 0 and z_n. Thus there exist constants b_n for $0 \leq n \leq N$ such that

$$\begin{aligned}\int_{\mathbb{D}} |G|^3 \overline{\operatorname{sgn} G} \widetilde{f}\, d\sigma &= \int_{\mathbb{D}} \mathcal{P}(|G|^3 \overline{\operatorname{sgn} G}) \widetilde{f}\, d\sigma \\ &= b_0 \widetilde{f}(0) + \sum_{n=1}^{N} b_n \widetilde{f}(z_n) \\ &= \frac{b_0}{G(0)} \left(f(0) - \sum_{j=1}^{N} f(z_j) h_j(0) \right) \\ &\quad + \sum_{n=1}^{N} b_n \frac{f'(z_n) - \sum_{j=1}^{N} f(z_j) h_j'(z_n)}{G'(z_n)}.\end{aligned}$$

But this is a linear combination of the numbers $f(0)$ and $f(z_n)$ and $f'(z_n)$, and thus I is a linear combination of $f(0)$ and $f(z_n)$ and $f'(z_n)$, where the specific linear combination depends on G and the numbers z_n.

Note that the term II is a linear combination of the numbers $f(z_n)$, and so both I and II are linear combinations of the numbers $f(0)$ and $f(z_n)$ and $f'(z_n)$. Thus G^2 is the kernel for a derivative evaluation functional depending on $f(0)$ and $f(z_n)$ and $f'(z_n)$. Thus, by Proposition 4.15 and Theorem 4.12, and the fact that $(A^p)^*$ and A^q are isomorphic, G^2 has the desired form.

We now proceed to the general case. Let $A_n = d_n((p/2) - 1)$. For $1 \leq n \leq N$ and $0 \leq j \leq A_n - 1$, let h_{nj} be a polynomial such that

$$h_{nj}^{(m)}(z_k) = \begin{cases} 1, & \text{if } m = n \text{ and } k = j \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in A^p$, define

$$\hat{f}(z) = f(z) - \sum_{n=1}^{N} \sum_{j=0}^{A_n - 1} a_{nj} h_{nj}(z)$$

where $a_{nj} = f^{(j)}(z_n)$. Since $\hat{f}$ has zeros of order $A_n = d_n((p/2) - 1)$ at each z_n, the function

$$\widetilde{f} = \frac{1}{G^{(p/2)-1}} \hat{f}$$

is in A^p.

But then

$$\begin{aligned}
\int_{\mathbb{D}} \overline{G}^{p/2} f \, d\sigma &= \int_{\mathbb{D}} \overline{G(z)}^{p/2} \left(\hat{f}(z) + \sum_{n=1}^{N} \sum_{j=0}^{A_n - 1} a_{nj} h_{nj}(z) \right) d\sigma \\
&= \int_{\mathbb{D}} |G(z)|^{p-1} \overline{\operatorname{sgn} G(z)} \widetilde{f}(z) \, d\sigma + \sum_{n=1}^{N} \sum_{j=0}^{A_n - 1} a_{nj} \int_{\mathbb{D}} \overline{G(z)}^{p/2} h_{nj}(z) \, d\sigma \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now, II is a linear combination of the numbers a_{nj} for $1 \leq n \leq N$ and $0 \leq j \leq A_n - 1$, so we turn our attention to I. The canonical divisor G is a constant multiple of the function $F \in A^p$ of smallest norm that has zeros of order d_n at each z_j and such that $F^{(m)}(0) = 1$, where m is the order of the zero-set at 0. Thus as before,

$$\int_{\mathbb{D}} |G|^{p-1} \overline{\operatorname{sgn} G} \widetilde{f} \, d\sigma = \int_{\mathbb{D}} \mathcal{P}(|G|^{p-1} \overline{\operatorname{sgn} G}) \widetilde{f} \, d\sigma = b_0 \widetilde{f}^{(m)}(0) + \sum_{n=1}^{N} \sum_{j=0}^{d_n - 1} b_{nj} \widetilde{f}^{(j)}(z_n),$$

for some complex constants b_0 and b_{nj}. Note that $\widetilde{f}(z) = G_n \hat{f}_n$ where

$$G_n(z) = (z - z_n)^{A_n} \frac{1}{G^{(p/2)-1}(z)}$$

and

$$\hat{f}_n(z) = (z - z_n)^{-A_n} \hat{f}(z).$$

Note that

$$\widetilde{f}^{(j)}(z_n) = \sum_{k=0}^{j} \binom{j}{k} \hat{f}_n^{(k)}(z_n) G_n^{(j-k)}(z_n)$$

and

$$\begin{aligned}
\hat{f}_n^{(k)}(z_n) &= \frac{k!}{(k + A_n)!} \frac{d^{k+A_n}}{dz^{k+A_n}} \hat{f}(z_n) \\
&= \frac{k!}{(k + A_n)!} \frac{d^{k+A_n}}{dz^{k+A_n}} \left[f(z) - \sum_{n=1}^{N} \sum_{s=0}^{A_n - 1} a_{ns} h_{ns}(z) \right]
\end{aligned}$$

Thus, $\widetilde{f}^{(j)}(z_n)$ is a linear function of the numbers a_{ns} and the numbers $f^{(k)}(z_n)$ for $0 \leq k \leq j + A_n$. Recall that $a_{ns} = f^{(s)}(z_n)$.

Also, if $m = 0$, then

$$\widetilde{f}^{(m)}(0) = G(0)^{1-(p/2)}\hat{f}(0) = G(0)^{1-(p/2)}\left(f(0) - \sum_{n=1}^{N}\sum_{j=0}^{A_n-1} a_{nj}h_{nj}(0)\right),$$

so $\widetilde{f}^{(m)}(0)$ is a linear function of the numbers a_{nj} and $f(0) = f^{(mp/2)}(0)$. If $m \neq 0$, then we may assume $z_1 = 0$ and $m = d_1$, and then by the same reasoning as we used above for $\widetilde{f}^{(j)}(z_n)$, we see that$\widetilde{f}^{(m)}(z_1)$ is a linear function of the numbers a_{nj} and the numbers $f^{(k)}(z_1)$ for $0 \leq k \leq d_1 + A_1 = d_1 + ((p/2) - 1)d_1 = (p/2)d_1 = mp/2$. Thus, term

$$\mathrm{I} = \int_{\mathbb{D}} \overline{G^{(p/2)-1}}\widetilde{f}\, d\sigma$$

is a linear combination of the numbers $f^{(k)}(z_n)$ for $0 \leq k \leq (d_n - 1) + ((p/2) - 1)d_n = (p/2)d_n - 1$, and the number $f^{(mp/2)}(0)$.

Therefore, both I and II, and thus $\int_{\mathbb{D}} f\overline{G}^{p/2}\, d\sigma$, are linear combinations of the numbers $f^{(k)}(z_n)$ for $0 \leq k \leq (p/2)d_n - 1$, and the number $f^{(mp/2)}(0)$. And thus, $G^{p/2}$ is the kernel for a derivative-evaluation functional depending on $f^{(j)}(z_n)$ for $1 \leq n \leq N$ and $0 \leq j \leq (p/2)d_n - 1$, as well as $f^{mp/2}(0)$. Therefore $G^{p/2}$ has the desired form. $\square$

The previous theorem gave a condition on $G^{p/2}$ that must be satisfied if G is the canonical divisor of a given zero set. The following theorem says that condition, along with a few other more obviously necessary ones, is also sufficient.

Theorem 4.29. *Let p be an even integer. Let $z_1, \ldots, z_N$ be distinct points in $\mathbb{D}$, and consider the zero-set consisting of each of these points with multiplicities $d_1, \ldots, d_N$, respectively, and let G be the canonical divisor for this zero set. Then G is the unique function having A^p norm 1 such that $G(0) > 0$ (or $G^{(m)}(0) > 0$ if G is required to have a zero of order m at the origin), and such that $G^{p/2}$ has zeros of order $pd_n/2$*

at each z_n, and

$$G(z)^{p/2} = c_0 + \sum_{n=1}^{N} \sum_{j=0}^{(p/2)d_n - 1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}} \qquad \textit{if } z_n \neq 0 \textit{ for all } n \textit{ or}$$

$$G(z)^{p/2} = c_0 z^{(p/2)d_1} + \sum_{j=0}^{(p/2)d_1 - 1} c_{1j} z^j + \sum_{n=2}^{N} \sum_{j=0}^{(p/2)d_n - 1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}} \qquad \textit{if } z_1 = 0.$$

Proof. By Theorem 4.28 and the definition of the canonical divisor, the stated conditions are necessary for a function to be the canonical divisor. Suppose that G is a function satisfying the stated conditions. We will prove the theorem by applying Theorem 4.4 to $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$.

Again, we will first prove the theorem for the case where $p = 4$, no z_n is zero, and all multiplicities are $d_n = 1$. By Proposition 4.11,

$$\mathcal{P}\left(\frac{1}{(1-\overline{z_n}z)^2}\,\overline{G(z)}\right) = 0$$

and

$$\mathcal{P}(\overline{G}) = \overline{G(0)},$$

and by Proposition 4.14

$$\mathcal{P}\left(\frac{1}{(1-\overline{z_n}z)^3}\,\overline{G(z)}\right) = \frac{\overline{z_n}}{2}\,\overline{G'(z_n)}\,\frac{1}{(1-\overline{z_n}z)^2}.$$

Now, by assumption G^2 is a linear combination of terms of the form 1, and $(1-\overline{z_n}z)^{-2}$, and $(1-\overline{z_n}z)^{-3}$, and so $\mathcal{P}(G^2\overline{G})$ is a linear combination of the function 1 and the functions $(1-\overline{z_n}z)^{-2}$ for $1 \leq n \leq N$. Thus G is a constant multiple of the canonical divisor, by Theorem 4.4, since some multiple of the canonical divisor satisfies the minimal interpolation problem of finding f with $f(0) = 1$ and $f(z_n) = 0$ and $\|f\|_{A^p}$ as small as possible. But the conditions $\|G\|_{A^p} = 1$ and $G(0) > 0$ imply that G must be the canonical divisor.

We will now discuss the general case under the assumption that $z_n \neq 0$ for all n. First, as above, $\mathcal{P}\left(\overline{G}^{(p/2)-1}\right) = \overline{G(0)}^{(p/2)-1}$. Now, by Proposition 4.17,

$$\mathcal{P}\left(\frac{1}{(1-\overline{z_n}z)^{j+2}}\,\overline{G(z)}^{(p/2)-1}\right) = \sum_{k=0}^{j-((p/2)-1)d_n} C_{n,j,k}\frac{1}{(1-\overline{z_n}z)^{k+2}},$$

where the constants $C_{n,j,k}$ may depend on G. But if $j \le (p/2)d_n - 1$, then $j - ((p/2)-1)d_n \le d_n - 1$. Thus

$$\mathcal{P}\left(G^{p/2}\,\overline{G(z)}^{(p/2)-1}\right) = B_0 + \sum_{n=1}^{N}\sum_{k=0}^{d_n-1}\frac{B_{n,k}}{(1-\overline{z_n}z)^{k+2}},$$

where $B_{n,k} = \sum_{j=k+((p/2)-1)d_n}^{(p/2)d_n-1} c_{nj}C_{n,j,k}$ and $B_0 = c_0\overline{G(0)}^{(p/2)-1}$. By Theorem 4.4, G is a multiple of the canonical divisor. But the conditions that $G^{(m)}(0) > 0$ and $\|G\|_{A^p} = 1$ imply that G is the canonical divisor.

The case where $z_1 = 0$ is similar, but we also use the fact that $\mathcal{P}(z^j\overline{G}^{(p/2)-1})$ is a polynomial of degree at most $j - [(p/2)-1]d_1$, or zero if $j < [(p/2)-1]d_1$. □

From previous work by MacGregor and Stessin [20], a weaker form of Theorem 4.28 is essentially known. In the weaker form of the theorem, one only knows, in the case that no $z_n = 0$, that

$$G(z) = c_0 + \sum_{n=0}^{N}\frac{b_n}{1-\overline{z_n}z} + \sum_{n=1}^{N}\sum_{j=0}^{(p/2)d_n-1}\frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}}$$

for some constants b_n. The case where $z_1 = 0$ is similar. To derive Theorem 4.29 from the weaker form of the theorem, we can use the following proposition.

This proposition also gives another indication of why the residues of $G^{p/2}$ must all be zero. It basically says that nonzero residues would lead to terms in $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$ that were kernels of functionals of the general form

$$f \mapsto \frac{1}{a}\int_0^a f(z)g(z)\,dz$$

where g is an analytic function and $a \in \mathbb{D}$. But, as the proposition explains, it would then be impossible for $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$ to be the kernel of a finite linear combination of derivative-evaluation functionals.

Proposition 4.30. *Let g be analytic on $\overline{\mathbb{D}}$ and suppose g is non-zero on $\partial\mathbb{D}$. Let $a_n \in \mathbb{D}$ and $a_n \neq 0$ for $1 \leq n \leq N$, and assume that $a_n \neq a_j$ for $n \neq j$. Let $b_n \in \mathbb{C}$ for $1 \leq n \leq N$. Then if any of the b_n are nonzero,*

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{g(z)}\right)$$

is not the kernel for a functional that is the finite linear combination of derivative-evaluation functionals.

Note that as is shown in [9] (see also [6] and [24]), the canonical divisor of a finite zero set is analytic in $\overline{\mathbb{D}}$ and non-zero on $\partial\mathbb{D}$. This allows the proposition to be applied to Bergman projections of the form

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{G(z)^{(p/2)-1}}\right).$$

Proof. We know by Proposition 4.18 that

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{g(z)}\right)$$

is the kernel for the functional given by

$$f \mapsto \sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} f(z)g(z)\,dz.$$

Suppose that this functional were a linear combination of derivative-evaluation functionals, which we will denote by $f \mapsto f^{(k)}(z_j)$, where $1 \leq j \leq J$ and $0 \leq k \leq K$. Let h be a function such that $h = gf$ for some $f \in A^p$. For fixed g, the values $f^{(k)}(z_j)$ for $1 \leq j \leq J$ and $0 \leq k \leq K$ are linear combinations of the values $h^{(k)}(z_j)$, where

$1 \leq j \leq J$ and $0 \leq k \leq K + r(z_j)$, and $r(z_j)$ is the order of the zero of g at z_j. Thus the functional defined on the space gA^p by

$$h \mapsto \sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} h(z)\, dz$$

must be a linear function of the values $h^{(k)}(z_j)$. By gA^p, we mean the vector space of all functions that may be written as g multiplied by an A^p function. Since g is analytic in $\overline{\mathbb{D}}$ and g is nonzero on $\partial\mathbb{D}$, any polynomial that has all the zeros of g will be in gA^p.

Now for each m there exists a polynomial H_m such that $H_m(a_m) = 1$, but $H_m(a_n) = 0$ for all $n \neq m$, and such that $H_m^{(k)}(z_j) = 0$ for all j and k such that $1 \leq j \leq J$ and $1 \leq k \leq K + r(z_j) + 1$. Also, we may require that H_m' has all the zeros of g, and that $H_m(0) = 0$. Set $h_m = H_m'$. Then h_m shares all the zeros of g, and so it is a multiple of g. Thus

$$\sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} h_m(z)\, dz = 0,$$

since the left side of the above equation is a linear combination of the numbers $h_m^{(k)}(z_j)$ for $1 \leq j \leq J$ and $0 \leq k \leq K + r(z_j)$, and each $h_m^{(k)}(z_j) = 0$. But also, for each m such that $1 \leq m \leq N$, we have

$$\sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} h_m(z)\, dz = \sum_{n=1}^{N} \frac{b_n}{a_n} H_m(a_n) = \frac{b_m}{a_m},$$

so each $b_m = 0$. $\square$

BIBLIOGRAPHY

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[1] D. Aharonov, C. Bénéteau, D. Khavinson, and H.S. Shapiro. Extremal problems for nonvanishing functions in Bergman spaces. In *Selected Topics in Complex Analysis*, Volume 158 of *Oper. Theory Adv. Appl.*, pages 59–86. Birkhäuser, Basel, 2005.

[2] D. Aharonov, H. S. Shapiro, and A. Yu. Solynin. A minimal area problem in conformal mapping II. *Journal d'Analyse Mathématique*, 83:259–288, 2001.

[3] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.

[4] J. B. Conway. *A Course in Functional Analysis*. Springer-Verlag, New York, second edition, 1990.

[5] P. Duren. *Theory of H^p spaces*. Dover Publications, Mineola, New York, 2000.

[6] P. Duren, D. Khavinson, and H. S. Shapiro. Extremal functions in invariant subspaces of Bergman spaces. *Illinois J. Math.*, 40(2):202–210, 1996.

[7] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg. Invariant subspaces in Bergman spaces and the biharmonic equation. *Michigan Math. J.*, 41(2):247–259, 1994.

[8] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg. Contractive zero-divisors in Bergman spaces. *Pacific J. Math.*, 157(1):37–56, 1993.

[9] P. Duren and A. Schuster. *Bergman spaces*. American Mathematical Society, Providence, RI, 2004.

[10] T. Ferguson. Continuity of Extremal Elements in Uniformly Convex Spaces. *Proc. Amer. Math. Soc.*, 137(8):2645–2653, 2009.

[11] J B. Garnett. *Bounded Analytic Functions*. Academic Press Inc., New York, 1981.

[12] J. Hansbo. Reproducing Kernels and Contractive Divisors in Bergman Spaces. *J. Math. Sci. (New York)*. 92(1):3657–3674, 1998. [Translated from Russian. Originally published in *Zapiski Nauchnykh Seminarov POMI*. 232:174–198, 1996.]

[13] H. Hedenmalm. A factorization theorem for square area-integrable analytic functions. *J. Reine Angew. Math.*, 422:45–68, 1991.

[14] E. Hewitt and K. Stromberg. *Real and Abstract Analysis*. Springer-Verlag, New York, 1975.

[15] C. Horowitz. Zeros of functions in the Bergman spaces. *Bull. Amer. Math. Soc.*, 80:713–714, 1974.

[16] C. Horowitz. Zeros of functions in the Bergman spaces. *Duke Math. J.*, 41:693–710, 1974.

[17] D. Khavinson, J. E. McCarthy, and H. S. Shapiro. Best approximation in the mean by analytic and harmonic functions. *Indiana Univ. Math. J.*, 49(4):1481–1513, 2000.

[18] D. Khavinson and M. Stessin. Certain linear extremal problems in Bergman spaces of analytic functions. *Indiana Univ. Math. J.*, 46(3):933–974, 1997.

[19] G. Köthe. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling, *Die Grundlehren der mathematischen Wissenschaften*, Band 159. Springer-Verlag New York Inc., New York, 1969.

[20] T. H. MacGregor and M. I. Stessin. Weighted reproducing kernels in Bergman spaces. *Michigan Math. J.*, 41(3):523–533, 1994.

[21] K. Yu. Osipenko and M. I. Stessin. On optimal recovery of a holomorphic function in the unit ball of $\mathbf{C}^n$. *Constr. Approx.*, 8(2):141–159, 1992.

[22] V. G. Ryabykh. Extremal problems for summable analytic functions. *Sibirsk. Mat. Zh.*, 27(3):212–217, 226, 1986. [In Russian].

[23] H. S. Shapiro. *Topics in approximation theory.* Springer-Verlag, Berlin, 1971.

[24] C. Sundberg. Analytic continuability of Bergman inner functions. *Michigan Math. J.*, 44(2):399–407, 1997.

[25] D. Vukotić. A sharp estimate for A^p_α functions in $\mathbf{C}^n$. *Proc. Amer. Math. Soc.*, 117(3):753–756, 1993.

[26] D. Vukotić. Linear extremal problems for Bergman spaces. *Exposition. Math.*, 14(4):313–352, 1996.

[27] A. Zygmund. *Trigonometric Series: Vols. I, II.* Second edition. Cambridge University Press, London, 1968.