Tim Ferguson - Research Statement

My main area of research interest involves spaces of analytic functions and their relations to operator theory and harmonic analysis. For example, I am interested in Hardy spaces, Bergman spaces, Fock spaces, and operators and linear functionals on these spaces. I am also interested in how these spaces relate to topics in harmonic analysis, including real valued Hardy spaces, Littlewood-Paley theory, maximal functions, and square functions.

Much of my work relates to extremal problems. A typical problem is as follows: suppose that ϕ is a bounded linear functional defined on some normed space of analytic functions. Maximize (the real part of) $\phi(f)$ over all functions in the space of unit norm. For example, we could try to maximize the real part of $f(0)+f(1/2)+f'(i/3)$ over all analytic functions in the unit disc that are bounded in absolute value by 1. Or we could maximize the same quantity over all functions analytic in the unit disc whose L^p norm (taken over the unit disc) is bounded by one.

I am also interested in finding upper and lower bounds for various operators on spaces of analytic function. For example, some coauthors and I have analyzed self commutators of Toeplitz operators in the Bergman space A^2. For Hardy spaces, the norms of such operators are connected with the isoperimetric inequality, which is quite surprising! It turns out that these operators on Bergman spaces are connected with Saint-Venant's inequality for torsional rigidity.

I will now describe my work in more detail.

Background on Spaces of Analytic Functions

The unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ is defined by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a function f analytic in the unit disc, its integral mean of radius r is defined by

$$M_p(r,f) = \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p},$$

when $0 < p < \infty$ and $0 \leq r < 1$. We also define $M_\infty(r,f) = \max_{0\leq\theta<2\pi} |f(re^{i\theta})|$. It can be seen that the integral means are non-decreasing as $r \to 1$. A function f is in the Hardy space H^p if $\sup_{0<r\leq 1} M_p(r,f) = \|f\|_{H^p} < \infty$. For $1 \leq p \leq \infty$, the Hardy space H^p is a Banach space with norm $\|\cdot\|_{H^p}$.

Any Hardy space function f can be factored into the form Bg, where B is a type of function known as a Blaschke product, and thus in H^∞, and g is an analytic function without zeros in the unit disc such that $\|f\|_{H^p} = \|g\|_{H^p}$. Such a factorization is extremely useful in many problems involving Hardy spaces, as it sometimes allows one to reduce consideration to the case of non-vanishing functions and thus apply results directly from the H^2 case, which is often simpler. The results stated above may be found in [3].

Hardy spaces arise naturally in many fields. The space H^∞ is just the space of bounded analytic functions in the unit disc, and so arises frequently. The Hilbert space H^2 arises naturally in operator theory. The Hardy spaces for $0 < p < 1$, and their generalizations to functions on $\mathbb{R}^n$ for $n \geq 2$, arise in harmonic analysis as substitutes for the L^p spaces, which are poorly behaved for $0 < p < 1$. Hardy space techniques are used in control theory, and extremal problems in Hardy spaces have applications in systems theory and operator theory.

Closely related to Hardy spaces are the Bergman spaces, which are denoted by A^p for $0 < p < \infty$. The Hilbert space A^2 was originally studied by Stefan Bergman, and his work was later generalized to include the A^p spaces for $0 < p < \infty$. A function f analytic in the unit disc is in the space A^p if

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} |f|^p \, d\sigma \right\}^{1/p} < \infty,$$

where σ denotes normalized Lebesgue area measure. It is not difficult to see that $H^p \subset A^p$ for all p such that $0 < p < \infty$. In studying Bergman spaces, known results for Hardy spaces are often taken as guides. However, Bergman spaces are generally less tractable than Hardy spaces. Some problems that are relatively simple for Hardy spaces, such as the description of zero-sets or invariant subspaces, are still open for Bergman spaces. It was only comparatively recently that "canonical divisors", which play a similar role in Bergman space as Blaschke products do in Hardy space, were discovered. In fact, they were discovered by investigating certain extremal problems in Bergman spaces (see [4]).

Background on Extremal Problems

Much of my current work centers on the theory of extremal problems in Bergman spaces and related spaces of analytic functions. The study of extremal functions in complex analysis is quite important, since they often have deep properties that are not immediately apparent from the extremal problem which they solve. Extremal problems in Hardy spaces have far reaching applications to diverse areas such as operator theory and prediction theory, and as extremal problems in Bergman spaces are studied further it can be expected that many applications will be found for them as well, especially since they are connected to certain partial differential equations, for example with the PDE defining p-harmonic functions.

The theory of extremal problems in Hardy spaces has been extensively studied, by S. Ya. Khavinson, A.J. Macintyre, W.W. Rogosinski, H.S. Shapiro, and others (see [3]). The study of extremal functions in Bergman spaces is much more difficult. For $1 < p < \infty$, the dual of the Bergman space A^p is isomorphic to A^q, where $1/p + 1/q = 1$. This isomorphism is not an isometry, but the norms are equivalent. Thus the typical extremal problem in A^p for $1 < p < \infty$ can be phrased as follows: given a functional $\phi \in (A^p)^*$ corresponding to a function $k \in A^q$, find a function $F \in A^p$ with $\|F\|_{A^p} = 1$ such that

$$\operatorname{Re} \phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re} \phi(g) = \operatorname{Re} \int_{\mathbb{D}} g\overline{k}\, d\sigma.$$

Because the Bergman spaces are uniformly convex for $1 < p < \infty$, a unique solution always exists in this case (see [5]). However, these problems are difficult to solve explicitly, and are much more difficult than the corresponding Hardy space problems.

My Research

Regularity for Extremal Problems

Much of my research focuses on a fascinating theorem of Ryabykh (see [10], [5]), which relates extremal problems in Bergman space to Hardy spaces. Suppose that $k \in H^q \subset A^q$, and define the functional $\phi \in (A^p)^*$ by

$$\phi(f) = \int_{\mathbb{D}} f(z)\overline{k(z)}\, d\sigma, \quad f \in A^p.$$

Then Ryabykh's theorem says that the extremal function over the space A^p is actually in H^p.

I was able to simplify and clarify Ryabykh's proof in the abstract context of uniformly convex Banach spaces, and to provide a bound on the H^p norm of the extremal function in terms of p and the A^q and H^q norms of k. I was also able to extend Ryabykh's theorem in the case where p is an even integer. The result I found is as follows: let $k \in A^q$ and let F be the A^p extremal function associated with the functional in $(A^p)^*$ whose kernel is k. Let p_1 and q_1 be a pair of numbers such that $q \leq q_1 < \infty$ and $p_1 = (p-1)q_1$. Then $F \in H^{p_1}$ if and only if $k \in H^{q_1}$. Also,

$$C_1 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)} \leq \|F\|_{H^{p_1}} \leq C_2 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)},$$

where C_1 and C_2 are constants depending only on p and p_1. For p an even integer, this theorem contains the converse of Ryabykh's theorem as a special case. This converse was previously unknown. I have also obtained a result which states that $F \in H^\infty$ if the coefficients of the Taylor expansion of k satisfy certain growth conditions. Recently, by using regularity results found in [8], I have been able to extend most of these results to the case where $1 < p < \infty$. The exception is the result saying that if $F \in H^{p_1}$, then $k \in H^{q_1}$.

Ryabykh proved his result by studying the integral $\int_0^{2\pi} |F(e^{i\theta})|^p \, d\theta$, where F is the extremal function. To prove the above results, I extended his methods to study the integral $\int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) \, d\theta$, where h can be any sufficiently well behaved function. Littlewood-Paley theory was key to handling certain technical issues arising in the study of the integral.

Recently, I have found results similar to Ryabykh's theorem for extremal problems in certain weighted Bergman spaces and Fock type spaces. For each fixed p such that $0 < p < \infty$, and each fixed $\alpha > 0$, the Fock space F^p_α is the space of all entire functions such that

$$\left\{ \int_{\mathbb{C}} |f(z)|^p e^{-\alpha p |z|^2} \, dA(z) < \infty. \right\}^{1/p}$$

The left-hand side of the above displayed inequality defines a norm for the Fock space. Define a Fock type space to be the space of all entire functions such that

$$\left\{ \int_{\mathbb{C}} |f(z)|^p \nu(z) \, dA(z) < \infty, \right\}^{1/p}$$

where ν is an appropriate non-negative weight function.

To state the results, let $\mathbb{D}_R$ denote the disc of radius R centered at the origin, and let $D_\infty = \mathbb{C}$. Given a weight function ν, let $A^p_R(\nu)$ be the space of analytic functions with norm

$$\left\{ \int_{\mathbb{D}_R} |f(z)|^p \nu(z)\, dA(z) \right\}^{1/p}.$$

With appropriate choice of ν, this is a Banach space. Suppose $\nu(z) = \omega(|z|^2)$, where ω is a positive decreasing function that is analytic on some open set containing $[0, R^2)$. For $k \in A^q_R(\nu)$, where q is the conjugate exponent to p, consider the extremal problem of finding a function F such that $\|F\|_{A^p_R(\nu)} = 1$ and

$$\operatorname{Re} \int_{\mathbb{D}_R} F(z)\overline{k(z)}\, \nu(z)\, d\sigma$$

is as large as possible. If the integral means of k grow sufficiently slowly, and if k is in the space $A^q_R(-|z|^2\omega'(|z|^2))$, then the integral means of F cannot grow too quickly, and F is also in the space $A^p_R(-|z|^2\omega'(|z|^2))$.

Explicit Solutions of Extremal Problems

I have also developed a method for finding explicit solutions to extremal problems in Bergman spaces. I have found explicit formulas for certain extremal functions of the type discussed above, and explicit formulas for the extremal functions for certain minimal interpolation problems, which are closely related to the type of extremal problem already discussed. Before my work, the only broadly applicable method of finding explicit extremal functions was the “guess and check” method, whereby one would make a guess at what the extremal function might be, then check it using some condition sufficient for a function to be an extremal function.

The methods I developed give new information about canonical divisors, an important type of function in Bergman spaces. Canonical divisors play a similar role to the Blaschke products in Hardy space theory. One can divide any function in A^p by the canonical divisor corresponding to its zero set to obtain a non-zero function with A^p norm less than or equal to the A^p norm of the original function.

Self-Commutators of Toeplitz Operators

A problem I have worked on that is somewhat related to my other work involves finding lower bounds for norms of self commutators of Toeplitz operators in the Bergman space. Some similar techniques are used to solve this problem and others I have worked on, and finding norms of operators is an extremal problem itself. Let $\mathcal{P}$ denote the Bergman projection. For a function ϕ, analytic and bounded in the disc, we define the Toeplitz operator T_ϕ by $T_\phi(f) = \mathcal{P}(\phi f) = \phi f$, for f in the Bergman space $A^2(G)$, where G is the domain under study. Then the adjoint of the operator is given by $T^*_\phi(f) = T_{\overline{\phi}}(f) = \mathcal{P}(\overline{\phi} f)$. My coauthors and I studied the norm of this operator in [2]. A similar study had been carried out in the Hardy space case by Dmitry Khavinson [7]. In this case, a sharp upper bound of the norm can be readily found from Putnam's inequality for hyponormal operators. A lower bound was found by Khavinson. Remarkably, Khavinson's inequality and Putnam's inequality together imply the two dimensional isoperimetric inequality.

Inspired by this result, we studied lower bounds of the norms of these self commutators for the Bergman space. We obtained two inequalities, one involving the fundamental frequency of the domain G, and the other involving its torsional rigidity. The torsional rigidity for a domain is the solution to a certain extremal problem, and it measures how hard it is to twist a beam that has a cross section in the shape of the given domain. The Saint-Venant inequality inequality states that among all simply connected domains with given area, the circle is the one having greatest torsional rigidity. We were able to use our bounds and Putnam's inequality to recover the Saint-Venant inequality, but with a non-sharp constant. We conjectured that our bounds were sharp, but that Putnam's inequality was not sharp in the case we were studying (although it is sharp in general). In [9], our conjecture was proved to be correct. Thus, our paper, along with [9], gives a new proof of the Saint-Venant inequality.

Bounds on Bergman Projections

I have also done work on various bounds for the Bergman projection. My results imply the well known result that the Bergman projection is bounded from L^p to A^p for $1 < p < \infty$. However, I have also obtained more precise results relating integral means of functions with the integral means of their Bergman projections. These bounds are related to the study of extremal problems, since if F is an extremal function, then $\mathcal{P}(|F|^p/\overline{F})$ is an integral kernel for which F is the extremal function. Thus, knowledge of bounds of the Bergman projection give knowledge of how certain bounds on extremal functions affect the corresponding integral kernels.

Works Cited

[1] Bell, Steven, Timothy Ferguson, Erik Lundberg. *Self-commutators of Toeplitz operators and isoperimetric inequalities*, Mathematical Proceedings of the Royal Irish Academy, (in press).

[2] Peter Duren. *Theory of H^p Spaces.* Dover, Mineola, New York, 2000.

[3] Peter Duren and Alexander Schuster. *Bergman Spaces.* American Mathematical Society, Providence, RI, 2004.

[4] Timothy Ferguson. *Continuity of extremal elements in uniformly convex spaces.* Proc. Amer. Math. Soc. 137 (2009), no. 8, 2645–2653.

[5] Timothy Ferguson, *Extremal problems in Bergman spaces and an extension of Ryabykh's theorem*, Illinois J. Math. 55 (2011), no. 2, 555–573 (2012).

[6] D. Khavinson, *A note on Toeplitz operators*, Lecture Notes in Math., vol. 1166, Geometry of Banach spaces, N. Kalton and E. Saab (eds.), 1984, pp. 89-95, Springer-Verlag, 1985.

[7] Dmitry Khavinson and Michael Stessin. *Certain linear extremal problems in Bergman spaces of analytic functions.* Indiana Univ. Math. J. 46 (1997), no. 3, 933–974.

[8] J-F. Olsen, M. C. Reguera, *On a sharp estimate for Hankel operators and Pu tnams inequality*, arXiv:1305.5193.

[9] V. G. Ryabykh. *Extremal problems for summable analytic functions.* Sibirsk. Mat. Zh. 27 (1986), no. 3, 212–217, 226.

[10] Dragan Vukotić. *Linear extremal problems for Bergman spaces.* Exposition. Math. 14 (1996), no. 4, 313–352.

[11] Kehe Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, vol. 263, Springer, New York, 2012.