

**EXTREMAL PROBLEMS IN BERGMAN SPACES AND AN
EXTENSION OF RYABYKH'S HARDY SPACE REGULARITY
THEOREM FOR $1 < p < \infty$**

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ABSTRACT. We study linear extremal problems in the Bergman space A^p of the unit disc, where $1 < p < \infty$. Given a functional on the dual space of A^p with representing kernel $k \in A^q$, where $1/p + 1/q = 1$, we show that if $q \leq q_1 < \infty$ and $k \in H^{q_1}$, then $F \in H^{(p-1)q_1}$. This result was previously known only in the case where p is an even integer. We also discuss related results.

An analytic function f in the unit disc \mathbb{D} belongs to the Bergman space A^p if

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} |f(z)|^p d\sigma(z) \right\}^{1/p} < \infty,$$

where σ is normalized area measure (so that $\sigma(\mathbb{D}) = 1$). For $1 < p < \infty$, each functional $\phi \in (A^p)^*$ can be uniquely represented by

$$\phi(f) = \int_{\mathbb{D}} f \bar{k} d\sigma$$

for some $k \in A^q$ (called the kernel of ϕ), where $q = p/(p-1)$ is the conjugate index.

In this paper we study regularity results for the extremal problem of maximizing $\operatorname{Re} \phi(f)$ among all functions $f \in A^p$ of unit norm. An important regularity result is Ryabykh's theorem, which states that if the kernel is actually in the Hardy space H^q , then the extremal function must be in the Hardy space H^p (see [14] or [6] for a proof). In [7], the following extensions of Ryabykh's theorem are shown in the case where p is an even integer:

- For $q \leq q_1 < \infty$, the extremal function $F \in H^{(p-1)q_1}$ if the kernel $k \in H^{q_1}$ (if $q_1 = q$ this is Ryabykh's theorem).
- If the Taylor coefficients of k satisfy a certain bound, then $F \in H^\infty$.
- The map sending a kernel $k \in H^q$ to its extremal function $F \in A^p$ is a continuous map from $H^q \setminus \{0\}$ into H^p .
- For $q \leq q_1 < \infty$, if the extremal function $F \in H^{(p-1)q_1}$, then the kernel $k \in H^{q_1}$. (In fact, the proof in [7] shows that this result holds if $1 < q_1 < \infty$).

We show that the first two results above hold for all p such that $1 < p < \infty$. We also show a weaker form of the third result holds for $1 < p < \infty$, while a weaker form of the fourth holds if $2 \leq p < \infty$. It is an open problem whether the last two results hold in their strong forms for $1 < p < \infty$.

To overcome certain technical difficulties in the proof, we rely on regularity results from [12] for extremal functions with polynomial kernels. These results rely

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on regularity theorems for complex analogues of p -harmonic functions. Our paper also uses an inequality based on Littlewood-Paley theory that was proved in [7].

1. EXTREMAL PROBLEMS AND RYABYKH'S THEOREM

We now introduce the topic of the paper in more detail. (See [7] for a slightly more detailed introduction). If f is an analytic function, $S_n f$ denotes its n^{th} Taylor polynomial at the origin. We denote Lebesgue area measure by dA , and normalized area measure by $d\sigma$, so that $\sigma(\mathbb{D}) = 1$.

We recall some basic facts about Hardy and Bergman spaces. For proofs and further information, see [3] and [5]. Suppose that f is analytic in the unit disc. For $0 < p < \infty$ and $0 < r < 1$, the p^{th} integral mean of f at radius r is

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p},$$

whereas if $p = \infty$ it is

$$M_\infty(f, r) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|.$$

The integral means are increasing functions of r for fixed f and p . An analytic function f is in the Hardy space H^p if $M_p(f, r)$ is bounded. The radial limit $f(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exists for almost every θ if f is an H^p function. For $0 < p < \infty$, we have that $f(re^{i\theta})$ approaches the boundary function $f(e^{i\theta})$ in $L^p(d\theta)$ as $r \rightarrow 1^-$. Two H^p functions whose boundary values agree on some set of positive measure are identical. The space H^p is a Banach space with norm

$$\|f\|_{H^p} = \sup_r M_p(f, r) = \|f(e^{i\theta})\|_{L^p}.$$

Thus we can regard H^p as a subspace of $L^p(\mathbb{T})$, where \mathbb{T} denotes the unit circle. If $1 < p < \infty$, the space H^p is reflexive.

If $f \in H^p$ and $1 < p < \infty$, then $S_n f \rightarrow f$ in H^p as $n \rightarrow \infty$, where $S_n f$ is the n^{th} partial sum of the Taylor series for f centered at the origin. The Szegő projection S maps each function $f \in L^1(\mathbb{T})$ to an analytic function defined by

$$Sf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z} dt$$

for $|z| < 1$. It fixes H^1 functions and maps L^p boundedly onto H^p for $1 < p < \infty$.

If $f \in L^p$ for $1 < p < \infty$ and $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then $Sf(z) = \sum_{n=0}^{\infty} a_n z^n$.

For $1 < p < \infty$, the dual space $(A^p)^*$ is isomorphic to A^q , where $1/p + 1/q = 1$. A functional $\phi \in (A^p)^*$ corresponds to $k \in A^q$ if $\phi(f) = \int_{\mathbb{D}} f(z) \overline{k(z)} d\sigma(z)$. This correspondence is conjugate linear and does not preserve norms, but it is the case that

$$(1.1) \quad \|\phi\|_{(A^p)^*} \leq \|k\|_{A^q} \leq C_p \|\phi\|_{(A^p)^*},$$

where C_p is a constant depending only on p . It can be shown that $C_p \leq \pi \csc(\pi/p)$ (see [2] and the proof of Theorem 6 in Section 2.4 of [5]). As with Hardy spaces, if $f \in A^p$ for $1 < p < \infty$, then $S_n f \rightarrow f$ in A^p as $n \rightarrow \infty$.

In this paper the only Bergman spaces we consider are those with $1 < p < \infty$. For a given linear functional $\phi \in (A^p)^*$ such that $\phi \neq 0$, we study the extremal

problem of finding a function $F \in A^p$ with norm $\|F\|_{A^p} = 1$ such that

$$(1.2) \quad \operatorname{Re} \phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re} \phi(g) = \|\phi\|.$$

Such a function F is called an extremal function, and we say that F is an extremal function for a function $k \in A^q$ if F solves problem (1.2) for the functional ϕ with kernel k . Note that for $p = 2$ the extremal function is $F = k/\|k\|_{A^2}$.

For $1 < p < \infty$ an extremal function always exists and is unique, which follows from the uniform convexity of A^p . Also, for any function F of unit A^p norm, there is some k such that F solves (1.2) for the functional ϕ with kernel k , and such a k is unique up to a positive scalar multiple. Furthermore, one such k is given by $\mathcal{P}(|F|^p/\bar{F})$, where \mathcal{P} is the Bergman projection (see [6] and [8]).

This problem has been studied by many authors, e.g. in [4], [8], [9], [12], [13] and [16]. Regularity results for solutions to this and similar problems can be found in [6], [7], [11] and [14]. See also the survey [1].

Even though it is well known, we restate the Cauchy-Green theorem, which is an important tool in this paper.

Cauchy-Green Theorem. *If Ω is a region in the plane with piecewise smooth boundary and $f \in C^1(\bar{\Omega})$, then*

$$\frac{1}{2i} \int_{\partial\Omega} f(z) dz = \int_{\Omega} \frac{\partial}{\partial \bar{z}} f(z) dA(z),$$

where $\partial\Omega$ denotes the boundary of Ω .

The next result is an important characterization of extremal functions in A^p for $1 < p < \infty$ (see [15], p. 55). The last part of the theorem follows from the previous parts by a standard approximation argument.

Theorem A. *Let $1 < p < \infty$ and let $\phi \in (A^p)^*$. A function $F \in A^p$ with $\|F\|_{A^p} = 1$ satisfies*

$$\operatorname{Re} \phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re} \phi(g) = \|\phi\|$$

if and only if $\operatorname{Re} \phi(F) > 0$ and

$$\int_{\mathbb{D}} h |F|^{p-1} \overline{\operatorname{sgn} F} d\sigma = 0$$

for all $h \in A^p$ with $\phi(h) = 0$. If F satisfies the above conditions, then

$$\int_{\mathbb{D}} h |F|^{p-1} \overline{\operatorname{sgn} F} d\sigma = \frac{\phi(h)}{\|\phi\|}$$

for all $h \in A^p$. Furthermore, suppose that $\phi(f) = \int_{\mathbb{D}} f \bar{k} d\sigma$ for some $k \in H^\infty$, and that $F \in H^\infty$. Then

$$\int_{\mathbb{D}} h |F|^{p-1} \overline{\operatorname{sgn} F} d\sigma = \int_{\mathbb{D}} h \bar{k} d\sigma$$

for any function $h \in L^1$.

Ryabikh's theorem is a result for extremal problems in Bergman spaces that involves Hardy space regularity. It says that if the kernel for a linear functional is not only in A^q but also in H^q , then the extremal function is in H^p as well as A^q .

Ryabykh's Theorem. *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Suppose that $\phi \in (A^p)^*$ and $\phi(f) = \int_{\mathbb{D}} f \bar{k} d\sigma$ for some $k \in H^q$. Then the solution F to the extremal problem (1.2) belongs to H^p and satisfies*

$$(1.3) \quad \|F\|_{H^p} \leq \left\{ \left[\max(p-1, 1) \right] \frac{C_p \|k\|_{H^q}}{\|k\|_{A^q}} \right\}^{1/(p-1)},$$

where C_p is the constant in (1.1).

Ryabykh proved that $F \in H^p$ in [14]. The bound (1.3) was proved in [6] by a variant of Ryabykh's proof.

In [7], it is shown that if p is an even integer, then for $q \leq q_1 < \infty$ the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$. It is also shown that if the Taylor coefficients of k satisfy a certain bound then $F \in H^\infty$, and that the map sending a kernel $k \in H^q$ to its extremal function $F \in A^p$ is a continuous map from $H^q \setminus \{0\}$ into H^p . We show that some of these results hold for any p such that $1 < p < \infty$ and that the others hold in weaker forms. It is still an open problem whether the weaker results can be improved so that they correspond to the results from the case when p is an even integer.

We need the following lemma for technical reasons.

Lemma 1.1. *If k is a polynomial, then $F' \in A^r$ for some $r > 1$, and $F \in H^\infty$.*

This follows from Corollary 2.1 in [12]. See page 944 of that paper for a justification of the fact that $F' \in A^r$.

The next lemma is a simplified version of Lemma 1.2 from [7].

Lemma 1.2. *Suppose that $1 < p_1 < \infty$ and $1 < p_2, p_3 \leq \infty$, and also that*

$$1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Let $f_1 \in H^{p_1}$, $f_2 \in H^{p_2}$, and $f_3 \in H^{p_3}$. Suppose further that $f_1 f_2 f_3'$ is in A^1 . Then

$$\left| \int_{\mathbb{D}} \bar{f}_1 f_2 f_3' d\sigma \right| \leq C \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}}$$

where C depends only on p_1 and p_2 . Moreover, if $p_3 < \infty$ then

$$\int_{\mathbb{D}} \bar{f}_1 f_2 f_3' d\sigma = \lim_{n \rightarrow \infty} \int_{\mathbb{D}} \bar{f}_1 f_2 (S_n f_3)' d\sigma.$$

The assumption on $f_1 f_2 f_3'$ is not essential, but without it the integral on the left needs to be replaced by a principle value. In the next lemma, the notation $\|f\|_{A^\infty}$ means the L^∞ norm of f on the disc, which of course is equivalent to the H^∞ norm.

Lemma 1.3. *If $1 \leq p \leq \infty$ and f is an analytic function with derivative in A^p , then*

$$\|f\|_{A^{2p}} \leq \|f\|_{H^p} \leq \|f'\|_{A^p} + |f(0)|.$$

The first inequality holds if $f \in H^p$.

Proof. The first inequality in this statement is from [17], and actually holds for $0 < p \leq \infty$. To prove the second inequality for $1 \leq p < \infty$ note that if $f(0) = 0$ then

$$\|f\|_{H^p}^p = \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 f'(re^{i\theta}) e^{i\theta} dr \right|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f'(re^{i\theta})|^p dr d\theta$$

by Jensen's inequality. But by Fubini's theorem, the last displayed expression equals

$$\int_0^1 M_p^p(r, f') dr = \int_0^{1/2} M_p^p(r, f') + M_p^p(1-r, f') dr.$$

But the integrand in the last integral is less than or equal to

$$2rM_p^p(r, f') + 2(1-r)M_p^p(1-r, f')$$

since $M_p^p(r, f') \leq M_p^p(1-r, f')$. But this means that the last displayed integral is bounded above by

$$\int_0^1 M_p^p(r, f') 2r dr = \|f'\|_{A^p}^p.$$

If $f(0) \neq 0$ note that

$$\|f\|_{H^p} \leq \|f - f(0)\|_{H^p} + |f(0)| \leq \|f'\|_{A^p} + |f(0)|.$$

The proof of the second inequality in the case $p = \infty$ is even easier, since then $|f(e^{i\theta})| \leq \sup_{0 \leq r < 1} |f'(re^{i\theta})| + |f(0)|$ for each θ . \square

2. THE NORM-EQUALITY FOR POLYNOMIALS

Let $1 < p < \infty$ and let q be its conjugate exponent. Let $k \in H^q$ and let F be the extremal function in A^p for k . We will denote by ϕ the functional associated with k . Define K by

$$(2.1) \quad K(z) = \frac{1}{z} \int_0^z k(\zeta) d\zeta;$$

thus $(zK)' = k$. Note that $\|K\|_{H^q} \leq \|k\|_{H^q}$ (see [6], equation (4.2)).

The first result in this article corresponds to Theorem 2.1 in [7].

Theorem 2.1. *Let $1 < p < \infty$, let k be a polynomial that is not identically 0, and let $F \in A^p$ be the extremal function for k . Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} F \left[\left(\frac{p}{2}\right) h\bar{k} + \left(1 - \frac{p}{2}\right) (zh)' \bar{K} \right] d\theta$$

for every polynomial h .

The proof of this Theorem is very similar to the proof of Theorem 2.1 in [7]. However, the proof in [7] also works if k is any H^q function.

Proof. Note that $F' \in A^s$ for some $s > 1$. By Ryabikh's theorem, $F \in H^p$. Now,

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta = \lim_{r \rightarrow 1} \frac{i}{2\pi r^2} \int_{\partial(r\mathbb{D})} |F(z)|^p h(z) z d\bar{z},$$

where h is any polynomial. Apply the Cauchy-Green theorem and take the limit as $r \rightarrow 1$ to transform the right-hand side into

$$\frac{1}{\pi} \int_{\mathbb{D}} \left((zh)' F + \frac{p}{2} zhF' \right) |F|^{p-1} \overline{\operatorname{sgn} F} dA(z).$$

We may apply Theorem A to reduce the last expression to

$$(2.2) \quad \frac{1}{\pi \|\phi\|} \int_{\mathbb{D}} \left((zh)' F + \frac{p}{2} zhF' \right) \bar{k} dA(z).$$

To prepare for a reverse application of the Cauchy-Green theorem, we rewrite the integral in (2.2) as

$$\lim_{r \rightarrow 1} \frac{1}{\pi \|\phi\|} \int_{r\mathbb{D}} \left[\frac{\partial}{\partial \bar{z}} \{(zh)' Fz\bar{K}\} + \frac{p}{2} \frac{\partial}{\partial z} \{zhF\bar{k}\} - \frac{p}{2} \frac{\partial}{\partial \bar{z}} \{(zh)' Fz\bar{K}\} \right] dA(z).$$

Since F is in H^p and both k and K are in H^q , we may apply the Cauchy-Green theorem and take the limit as $r \rightarrow 1$ to see that the above expression equals

$$\begin{aligned} & \frac{1}{2\pi i \|\phi\|} \int_{\partial\mathbb{D}} (zh)' Fz\bar{K} dz + \frac{ip}{4\pi \|\phi\|} \int_{\partial\mathbb{D}} zhF\bar{k} d\bar{z} \\ & \quad - \frac{p}{4\pi i \|\phi\|} \int_{\partial\mathbb{D}} (zh)' Fz\bar{K} dz \\ & = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} \left[(zh)' F\bar{K} + \frac{p}{2} hF\bar{k} - \frac{p}{2} (zh)' F\bar{K} \right] d\theta. \end{aligned}$$

□

As in [7], taking $h = 1$ gives the following corollary, which we call the “norm-equality.”

Corollary 2.2. (The Norm-Equality). *Let $1 < p < \infty$, let k be a polynomial that is not identically 0, and let F be the extremal function for k . Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p d\theta = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} F \left[\left(\frac{p}{2}\right) \bar{k} + \left(1 - \frac{p}{2}\right) \bar{K} \right] d\theta.$$

We use the norm-equality to give the following theorem, which corresponds with Theorem 2.3 in [7]. Unfortunately, the theorem in this article is weaker, and it seems difficult to prove a statement as strong as the one in [7]. In the statement of the theorem, $F_n \rightarrow F$ means that F_n converges to F in the weak sense.

Theorem 2.3. *Let $\{k_n\}$ be a sequence of functions in $H^q \setminus \{0\}$ and let $k_n \rightarrow k$ in H^q , where k is not identically zero. Let F_n be the A^p extremal function for k_n and let F be the A^p extremal function for k . Then $F_n \rightarrow F$ in H^p . Furthermore, if k and all the k_n are polynomials, then $F \rightarrow F$ in H^p .*

Because the operator taking a kernel to its extremal function is not linear, one cannot automatically conclude that $F_n \rightarrow F$ just because the operator is bounded. It seems likely that $F_n \rightarrow F$ holds for any k_n and k in H^q such that $k_n \rightarrow k$, and not just for polynomials, but we do not know a proof of this.

Proof. The proof is basically identical to the corresponding proof in [7], but we will summarize it for the sake of completeness.

To see that $F_n \rightarrow F$ in H^p , note that if F_n did not approach F weakly in H^p , then since Ryabykh’s theorem implies that the sequence $\{F_n\}$ is bounded in H^p norm, the Banach-Alaoglu theorem and the reflexivity of H^p would imply that some subsequence would converge weakly, and thus pointwise, to a function not equal to F . But $k_n \rightarrow k$ in A^q , and it is proved in [6] that this implies $F_n \rightarrow F$ in A^p , which implies $F_n \rightarrow F$ pointwise, a contradiction.

If k and all the k_n are polynomials, then the fact that $F_n \rightharpoonup F$ together with the norm-equality implies that $\|F_n\|_{H^p} \rightarrow \|F\|_{H^p}$. Since H^p is uniformly convex, it follows from $F_n \rightharpoonup F$ and $\|F_n\|_{H^p} \rightarrow \|F\|_{H^p}$ that $F_n \rightarrow F$ in H^p . \square

3. FOURIER COEFFICIENTS OF $|F|^p$

We now give some results about the Fourier coefficients of $|F|^p$ that follow from Theorem 2.1. The first result gives information about the Fourier coefficients of $|F|^p$ for nonpositive indices. Since $|F|^p$ is real valued, it also indirectly gives information about the Fourier coefficients for positive indices.

Theorem 3.1. *Let $1 < p < \infty$. Let k be a polynomial (not the zero polynomial), let F be the A^p extremal function for k , and define K by equation (2.1). Then for any integer $m \geq 0$,*

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p e^{im\theta} d\theta = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} F e^{im\theta} \left[\left(\frac{p}{2}\right) \bar{k} + \left(1 - \frac{p}{2}\right) (m+1) \bar{K} \right] d\theta.$$

Proof. Take $h(e^{i\theta}) = e^{im\theta}$ in Theorem 2.1. \square

The next result is a bound on the Fourier coefficients of $|F|^p$.

Theorem 3.2. *Let $1 < p < \infty$. Let k be a polynomial that is not the zero polynomial, and let k have associated functional $\phi \in (A^p)^*$. Let F be the A^p extremal function for k . Define*

$$b_m = \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p e^{-im\theta} d\theta,$$

and let

$$k(z) = \sum_{n=0}^N c_n z^n.$$

Then, for each $m \geq 0$,

$$|b_m| = |b_{-m}| \leq \frac{p}{2\|\phi\|} \|F\|_{H^2} \left[\sum_{n=m}^N |c_n|^2 \right]^{1/2}.$$

The proof of the theorem is identical to the one found in [7], and thus will be omitted. An interesting observation is that this theorem implies that $|F|^p$ is a trigonometric polynomial of degree at most N .

The estimate in Theorem 3.2 can be used to obtain information about the size of $|F|^p$ (and thus of F), as in the following corollary.

Corollary 3.3. *If $k \in H^q \setminus \{0\}$ and if $c_n = O(n^{-\alpha})$ for some $\alpha > 3/2$, then $F \in H^\infty$.*

Proof. Assume first that k is a polynomial. Observe that for $m \geq 2$ we have

$$\sum_{n=m}^{\infty} (n^{-\alpha})^2 \leq \int_{m-1}^{\infty} x^{-2\alpha} dx = \frac{(m-1)^{1-2\alpha}}{2\alpha-1},$$

and thus

$$\left[\sum_{n=m}^{\infty} |c_n|^2 \right]^{1/2} \leq C \frac{(m-1)^{(1/2)-\alpha}}{\sqrt{2\alpha-1}},$$

where C is the constant implicit in the expression $O(n^{-\alpha})$. Thus we have (for $m \geq 2$) that

$$|b_m| = |b_{-m}| \leq C \frac{p}{2\|\phi\|} \|F\|_{H^2} \frac{(m-1)^{(1/2)-\alpha}}{\sqrt{2\alpha-1}}$$

Therefore,

$$\begin{aligned} \sum_{m=3}^{\infty} |b_{-m}| &= \sum_{m=3}^{\infty} |b_m| \leq C \frac{p}{2\|\phi\|} \|F\|_{H^2} \int_2^{\infty} \frac{(x-1)^{(1/2)-\alpha}}{\sqrt{2\alpha-1}} dx \\ &\leq C \frac{p}{2\|\phi\|} \|F\|_{H^2} \frac{1}{(\alpha-3/2)\sqrt{2\alpha-1}}. \end{aligned}$$

But this implies that

$$\|F\|_{H^\infty}^p = \| |F|^p \|_{L^\infty} \leq \sum_{n=-\infty}^{\infty} |b_n| \leq C \frac{p}{\|\phi\|} \|F\|_{H^2} \frac{1}{(\alpha-3/2)\sqrt{2\alpha-1}} + \sum_{m=-2}^2 |b_m|.$$

Since each $|b_m| \leq \|F\|_{H^p}^p < \infty$, we have that

$$\|F\|_{H^\infty}^p \leq C \frac{p}{\|\phi\|} \|F\|_{H^2} \frac{1}{(\alpha-3/2)\sqrt{2\alpha-1}} + 5\|F\|_{H^p}^p.$$

Since $\|F\|_{H^2} \leq \|F\|_{H^\infty} < \infty$, we have that

$$\|F\|_{H^\infty}^{p-1} \leq C \frac{p}{\|\phi\|} \frac{1}{(\alpha-3/2)\sqrt{2\alpha-1}} + 5\|F\|_{H^p}^p.$$

Here we have also used the fact that $\|F\|_{H^\infty}^{-1} \leq \|F\|_{A^p}^{-1} = 1$.

Now we drop the assumption that k is a polynomial. Let F_n be the extremal function for $S_n k$, and let ϕ_n be the corresponding functional. By Ryabykh's theorem and the fact that $S_n k \rightarrow k$ in H^q , the sequence $\|F_n\|_{H^p}$ is bounded. Now, the above displayed inequality holds with F_n in place of F and ϕ_n in place of ϕ , since C can be taken to be independent of n . Also, it follows from the fact that $S_n k \rightarrow k$ in A^q that $\phi_n \rightarrow \phi$ in $(A^p)^*$, and that $F_n \rightarrow F$ in A^p and thus uniformly on compact subsets. Therefore,

$$\|F\|_{H^\infty}^{p-1} \leq \liminf_{n \rightarrow \infty} \|F_n\|_{H^\infty}^{p-1} \leq C \frac{p}{\|\phi\|} \frac{1}{(\alpha-3/2)\sqrt{2\alpha-1}} + 5 \liminf_{n \rightarrow \infty} \|F_n\|_{H^p}^p.$$

This proves the result. \square

4. RELATIONS BETWEEN THE SIZE OF THE KERNEL AND EXTREMAL FUNCTION

In this section we show that if $1 < p < \infty$ and $q \leq q_1 < \infty$ and the kernel $k \in H^{q_1}$ then the extremal function $F \in H^{(p-1)q_1}$. For $q_1 = q$ the statement reduces to Ryabykh's theorem. For p an even integer, this statement and its converse are proved in [7]. It is still an open problem to decide if the converse holds for general p , although we prove a weaker result similar to it.

We first prove the following theorem.

Theorem 4.1. *Let $1 < p < \infty$ and let $q = p/(p-1)$ be its conjugate exponent. Let $F \in A^p$ be the extremal function corresponding to the kernel $k \in A^q$, where k is a polynomial. Let $p \leq p_1 < \infty$, and $q \leq q_1 < \infty$. Define p_2 by*

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Then for every trigonometric polynomial h we have

$$\left| \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta \right| \leq C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \|F\|_{H^{p_1}} \|h\|_{L^{p_2}},$$

where C is some constant depending only on p , p_1 , and q_1 .

Note that the case $p_2 = \infty$ occurs if and only if $q = q_1$ and $p = p_1$. The theorem is then a trivial consequence of Ryabykh's theorem, so we need only prove the theorem if $p_2 < \infty$.

Proof. Let h be an analytic polynomial. In the proof of Theorem 2.1, we showed that

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta = \frac{1}{\pi \|\phi\|} \int_{\mathbb{D}} \left((zh)'F + \frac{p}{2} zhF' \right) \bar{k} dA(z).$$

Apply Lemma 1.2 separately to the two parts of the integral to conclude that its absolute value is bounded by

$$C \frac{1}{\|\phi\|} \|k\|_{H^{q_1}} \|f\|_{H^{p_1}} \|h\|_{H^{p_2}},$$

where C is a constant depending only on p_1 and q_1 . Since

$$\frac{1}{\|\phi\|} \leq \frac{C_p}{\|k\|_{A^q}}$$

by equation (1.1), the desired result holds for the case where h is an analytic polynomial. If h is an arbitrary trigonometric polynomial, then as in [7] the boundedness of the Szegő projection can be used to show the result holds. \square

For a given $q_1 > q$, we will apply the theorem just proven with $p_1 = (p-1)q_1$ and with p'_2 chosen to equal p_1/p , where p'_2 is the conjugate exponent to p_2 . This allows us to bound the H^{p_1} norm of f in terms of $\|\phi\|$ and $\|k\|_{H^{q_1}}$ only.

Theorem 4.2. *Let $1 < p < \infty$, and let q be its conjugate exponent. Let $F \in A^p$ be the extremal function for a kernel $k \in A^q$. If for some q_1 such that $q \leq q_1 < \infty$ the kernel $k \in H^{q_1}$, then $F \in H^{p_1}$ for $p_1 = (p-1)q_1$. In fact,*

$$\|F\|_{H^{p_1}} \leq C \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)},$$

where C depends only on p and q_1 .

The proof of this theorem is identical to the proof of the corresponding theorem in [7], so we give a summary.

Proof. The case $q_1 = q$ is Ryabykh's theorem, so we assume $q_1 > q$. Let $p_1 = (p-1)q_1$; thus $p_1 > p = (p-1)q$. Let $p_2 = p_1/(p_1 - p)$, so

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1$$

and $p'_2 = p_1/p$ and $1 < p_2 < \infty$. Let F_n denote the extremal function corresponding to the kernel $S_n k$ (where we choose n large enough so that $S_n k$ is not identically zero). Then for any trigonometric polynomial h , Theorem 4.1 implies that

$$\left| \frac{1}{2\pi} \int_0^{2\pi} |F_n|^p h(e^{i\theta}) d\theta \right| \leq C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}} \|h\|_{L^{p_2}}.$$

Taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{p_2}} \leq 1$ gives

$$\|F_n\|_{H^{p_1}}^p = \| |F_n|^p \|_{L^{p_2'}} \leq C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}}.$$

Because $\|F_n\|_{H^{p_1}} < \infty$ (since $S_n k$ is a polynomial) we may divide both sides of the inequality by $\|F_n\|_{H^{p_1}}$ to obtain

$$\|F_n\|_{H^{p_1}}^{p-1} \leq C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}},$$

where C depends only on p and q_1 . Taking the limit as $n \rightarrow \infty$ gives the desired result. \square

Recall from Section 1 that if $F \in A^p$ has unit norm, there is a corresponding kernel $k \in A^q$ such that F is the extremal function for k , and that this kernel is uniquely determined up to a positive multiple. Thus, it makes sense to ask if the converse of Theorem 4.2 holds. That is, does $F \in H^{(p-1)q_1}$ imply that $k \in H^{q_1}$? If p is an even integer and $q \leq q_1 < \infty$ then by Theorem 4.3 in [7] this is the case. In fact, the proof in [7] works for any q_1 such that $1 < q_1 < \infty$ (as long as p is an even integer). For general p we do not know if the result is still true. The result does hold if $2 \leq p < \infty$ and $1 < q_1 < \infty$ and if F is nonvanishing, since the proof in [7] works in that case. For general F we can prove the following weaker result for $2 \leq p < \infty$.

Theorem 4.3. *Suppose $2 \leq p < \infty$ and $1 < q_1 < \infty$. Let $F \in A^p$ with $\|F\|_{A^p} = 1$, and let k be a kernel such that F is the extremal function for k . Let $p_1 = q_1(p-1)$ and let $p_2 = pq_1/(q_1+1)$. If $F \in H^{p_1}$ and $F' \in A^{p_2}$ then $k \in H^{q_1}$ and*

$$\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \leq \left[\csc\left(\frac{\pi}{p}\right) \right] \left(\|F\|_{H^{p_1}}^{p-1} + \frac{p-2}{2} (\|F'\|_{A^{p_2}} + |F(0)|)^{p-2} \|F'\|_{A^{p_2}} \right),$$

where C_p is as in inequality (1.1).

Proof. Note first that the case $p = 2$ is trivial since then F and k are constant multiples of each other, so assume $p \neq 2$. Let q denote the exponent conjugate to p . Let h be a polynomial and let ϕ be the functional in $(A^p)^*$ corresponding to k . Then by Theorem A and the Cauchy-Green theorem,

$$\begin{aligned} & \frac{1}{\|\phi\|} \int_{\mathbb{D}} \overline{k(z)} (zh(z))' d\sigma \\ &= \int_{\mathbb{D}} |F(z)|^{p-1} \operatorname{sgn}(\overline{F(z)}) (zh(z))' d\sigma \\ &= \lim_{r \rightarrow 1} \int_{r\mathbb{D}} \left\{ \frac{\partial}{\partial z} [|F|^{p-1} \operatorname{sgn} \overline{F} zh] - \frac{p-2}{2} |F|^{p-2} F' \operatorname{sgn} \overline{F}^2 zh \right\} \frac{dA}{\pi} \\ &= \lim_{r \rightarrow 1} \frac{i}{2\pi} \int_{\partial(r\mathbb{D})} |F|^{p-1} \operatorname{sgn} \overline{F} zh d\bar{z} - \int_{\mathbb{D}} \frac{p-2}{2} |F|^{p-2} F' \operatorname{sgn} \overline{F}^2 zh d\sigma \\ &= \frac{1}{2\pi} \int_0^{2\pi} |F|^{p-1} (\operatorname{sgn} \overline{F}) h d\theta - \int_{\mathbb{D}} \frac{p-2}{2} |F|^{p-2} F' \operatorname{sgn} \overline{F}^2 zh d\sigma. \end{aligned}$$

Here we have used the fact that $|F|^{p-2} F' \in L^1$, which follows from the fact that $(p-2)/p + 1/p_2 < 1$. Now apply Hölder's inequality to the first integral using exponents q_1 and $q_1' = q_1/(q_1-1)$, and apply it to the second using exponents

$2p_2/(p-2)$ and p_2 and $2q'_1$ to obtain that the above expression is bounded above in absolute value by

$$\|F\|_{H^{p_1}}^{p-1} \|h\|_{H^{q'_1}} + \frac{p-2}{2} \|F\|_{A^{2p_2}}^{p-2} \|F'\|_{A^{p_2}} \|h\|_{A^{2q'_1}}.$$

But by Lemma 1.3, this is at most

$$\left(\|F\|_{H^{p_1}}^{p-1} + \frac{p-2}{2} (\|F'\|_{A^{p_2}} + |F(0)|)^{p-2} \|F'\|_{A^{p_2}} \right) \|h\|_{H^{q'_1}}.$$

Let C equal the part of the above expression in parentheses. Then

$$\left| \int_{\mathbb{D}} \overline{k(z)} (zh(z))' d\sigma \right| \leq C \|\phi\| \|h\|_{H^{q'_1}}$$

for all polynomials h , and we can define a continuous linear functional ψ on $H^{q'_1}$ so that

$$\psi(h) = \int_{\mathbb{D}} \overline{k(z)} (zh(z))' d\sigma$$

for all polynomials h . Then ψ has an associated kernel in H^{q_1} (see p. 113 of [3]). Call the kernel \tilde{k} . For $h \in H^{q_1}$ it follows that

$$\psi(h) = \frac{1}{2\pi} \int_0^{2\pi} \overline{\tilde{k}(e^{i\theta})} h(e^{i\theta}) d\theta.$$

By the Cauchy-Green theorem,

$$\begin{aligned} \int_{\mathbb{D}} \overline{k(z)} (zh(z))' d\sigma &= \psi(h) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \overline{\tilde{k}(e^{i\theta})} h(e^{i\theta}) d\theta \\ (4.1) \quad &= \lim_{r \rightarrow 1} \frac{i}{2\pi} \int_{\partial(r\mathbb{D})} \overline{\tilde{k}(z)} h(z) z d\bar{z} \\ &= \lim_{r \rightarrow 1} \int_{r\mathbb{D}} \overline{\tilde{k}(z)} (zh(z))' \frac{dA}{\pi} \\ &= \int_{\mathbb{D}} \overline{\tilde{k}(z)} (zh(z))' d\sigma, \end{aligned}$$

where h is any polynomial.

Define the polynomial H by

$$H(z) = \frac{1}{z} \int_0^z h(\zeta) d\zeta.$$

Then substituting $H(z)$ for $h(z)$ in equation (4.1), and using the fact that $(zH)' = h$, we have

$$\int_{\mathbb{D}} \overline{\tilde{k}(z)} h(z) d\sigma = \int_{\mathbb{D}} \overline{\tilde{k}(z)} h(z) d\sigma$$

for every polynomial h . Since $k \in A^q$ and $\tilde{k} \in H^{q_1} \subset A^{2q_1}$, we have that the power series for k and \tilde{k} converge in A^q and A^{2q_1} respectively. Using this fact and choosing $h(z) = z^n$ for $n \in \mathbb{N}$ shows that the power series of k and \tilde{k} are identical, and so $k = \tilde{k}$ and $k \in H^{q_1}$.

Now for any polynomial h ,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \overline{k(e^{i\theta})} h(e^{i\theta}) d\theta \right| \leq C \|\phi\| \|h\|_{H^{q_1}} \leq C \|k\|_{A^q} \|h\|_{H^{q_1}},$$

where we have used inequality (1.1). But for any trigonometric polynomial h , we have

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^{2\pi} \overline{k(e^{i\theta})} h(\theta) d\theta \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \overline{k(e^{i\theta})} [S(h)(e^{i\theta})] d\theta \right| \\ &\leq C \|k\|_{A^q} \|S(h)\|_{H^{q_1}} \\ &\leq C \csc\left(\frac{\pi}{p}\right) \|k\|_{A^q} \|h\|_{L^{q_1}}, \end{aligned}$$

where S denotes the Szegő projection. Note that $\csc(\pi/p)$ is the norm of the Szegő projection on $L^p(\partial\mathbb{D})$ (see [10]). Now take the supremum over all trigonometric polynomials h with $\|h\|_{L^{q_1}} \leq 1$ and divide both sides of the inequality by $\|k\|_{A^q}$. \square

It is interesting to note that the value of p_2 in the above theorem is less than p no matter the value of q_1 .

5. OPEN PROBLEMS AND A SIMPLE RESULT

As we have noted, unlike in the case in which p is an even integer, we do not know how to show that if $F \in H^{(p-1)q_1}$ then $k \in H^{q_1}$. However, we can show that a corresponding result holds if we replace the Hardy spaces by Bergman spaces. This result is not difficult and may be well known, but we do not know of anywhere it appears in the literature.

Theorem 5.1. *Let $1 < p < \infty$. Suppose $k \in A^q$ and F is the A^p extremal function for k . If $F \in A^{(p-1)q_1}$ for $1 < q_1 < \infty$, then $k \in A^{q_1}$. If $F \in H^\infty$, then k is in the Bloch space, and if F is continuous on the closed disc, then k is in the little Bloch space.*

Proof. As stated above, k must be a positive scalar multiple of $\mathcal{P}(|F|^p/\overline{F}) = \mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$, where \mathcal{P} is the Bergman projection. The result now follows since the Bergman projection is bounded from L^r to A^r for $1 < r < \infty$, and since it maps L^∞ onto the Bloch space and the space of continuous functions on the closed disc onto the little Bloch space (see e.g. [5]). \square

We now mention some open problems that could motivate further study.

- (1) For $1 < p < \infty$, if $F \in H^{(p-1)q_1}$, is $k \in H^{q_1}$? As we have said, this is known from [7] to be true if p is an even integer, or if F is nonvanishing and $2 \leq p < \infty$.
- (2) Is it the case that if $k \in A^{q_1}$, where $1 < q_1 < \infty$, then F must be in $A^{(p-1)q_1}$? If not, can anything interesting be said about the regularity of F ?
- (3) If k is in the Bloch space or the little Bloch space, can anything of interest be said about the regularity of F ?
- (4) If $k \in H^\infty$, must $F \in \operatorname{BMO}$? If $F \in H^\infty$, must $k \in \operatorname{BMO}$?
- (5) Does the generalization of Ryabykh's theorem (Theorem 4.2) hold for $1 < q_1 < q$?

- (6) Is the mapping from kernels to Bergman space extremal functions continuous on Hardy spaces? Is the mapping from extremal functions to kernels continuous on Hardy spaces? (Of course, there are multiple kernels with the same extremal function, but they are all positive scalar multiples of each other, so one can make sense of this question by specifying which kernel is chosen).

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