Bounds on Integral Means of Bergman Projections and their Derivatives

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Abstract

We bound integral means of the Bergman projection of a function in terms of integral means of the original function. As an application of these results, we bound certain weighted Bergman space norms of derivatives of Bergman projections in terms of weighted L^p norms of certain derivatives of the original function in the θ direction. These results easily imply the well known result that the Bergman projection is bounded from the Sobolev space $W^{k,p}$ into itself for $1 < p < \infty$. We also apply our results to derive certain regularity results involving extremal problems in Bergman spaces. Lastly, we construct a function that approaches 0 uniformly at the boundary of the unit disc but whose Bergman projection is not in H^2.

For $0 < p < \infty$, the Bergman space $A^p = A^p(\mathbb{D})$ is the space of all analytic functions in the unit disc $\mathbb{D}$ such that

$$\|f\|_{A^p} = \left[\int_{\mathbb{D}} |f(z)|^p d\sigma(z)\right]^{1/p} < \infty.$$

Here, σ is normalized Lebesgue area measure, so that $\sigma(\mathbb{D}) = 1$. The Bergman spaces are closed subspaces of $L^p(\mathbb{D})$ (see [4] or [8]).

For a function in L^p for $0 < p < \infty$, we define its p^{th} integral mean at radius r by

$$M_p(r, f) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right]^{1/p}.$$

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If $p = \infty$, we can define $M_p(r, f) = \operatorname{ess\,sup}_{0\le\theta<2\pi} |f(re^{i\theta})|$. It is well known that if f is analytic, then the integral means are nondecreasing functions of r (see [3]).

For $0 < p \le \infty$, the Hardy space consists of all analytic functions in $\mathbb{D}$ for which $\|f\|_{H^p} = \sup_{0\le r<1} M_p(r, f) < \infty$. It is easy to see that $H^p \subset A^p$ for $0 < p < \infty$. In fact, $H^{2p} \subset A^p$, and the H^{2p} norm is always greater than or equal to the A^p norm (see [14]), a fact which is related to the isoperimetric inequality.

If $f \in L^1(\mathbb{D})$, we define its Bergman projection to be

$$\mathcal{P}f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w)$$

for $z \in \mathbb{D}$. The function $\mathcal{P}(f)$ is an analytic function in the unit disc. When restricted to $L^2(\mathbb{D})$, the Bergman projection is the orthogonal projection onto $A^2(\mathbb{D})$. It is well known that the Bergman projection is bounded for $1 < p < \infty$ (see [4] or [8]). Our methods allow us to obtain the bound $2\pi/\sin(\pi/p)$ for the norm of the Bergman projection from L^p to L^p. We remark that it is known that the norm is at least $1/(2\sin(\pi/p))$, and at most $\pi/\sin(\pi/p)$, so that our bound differs from the norm by a factor that is between $1/(4\pi)$ and $1/2$ for each p (see [2]).

The main result of this article bounds integral means of derivatives of the Bergman projection of a function in terms of integral means of angular derivatives of the original function. These bounds are then used to bound certain weighted Bergman space norms of derivatives of the Bergman projection of a function in terms of certain weighted L^p norms of derivatives of the original function in the θ direction. (See the articles [9] and [10] for similar results in the context of several complex variables.) Our results easily imply the well known result that $\mathcal{P}$ is bounded from the Sobolev space $W^{k,p}$ into itself for $1 < p < \infty$, where k is a nonnegative integer. Lastly, we give a result in the opposite direction from our main result: there exists a function f such that $f(re^{i\theta}) \to 0$ uniformly as $r \to 1^-$, but for which the integral means $M_2(r, \mathcal{P}f)$ are not bounded in r.

It may seem unusual to investigate integral means of Bergman projections, since integral means are related to Hardy spaces and the Bergman projection is related to Bergman spaces. Therefore we give some motivation. In [13], Ryabykh found a relation between Hardy spaces and extremal problems in Bergman spaces. More specifically, he proved the following theorem: Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Suppose that $\phi \in (A^p)^*$ and that

$\phi(f) = \int_{\mathbb{D}} f\overline{k}\, d\sigma$ for some $k \in H^q$, where $k \neq 0$. (The function k is called the integral kernel of ϕ.) Then the solution to the extremal problem of finding the function $F \in A^p$ of unit norm that maximizes $\operatorname{Re}\phi(F)$ belongs to H^p. (It is known that such an F is unique.) Also, F satisfies the bound

$$\|F\|_{H^p} \leq \left\{ \left[\max(p-1,1) \right] \frac{C_p \|k\|_{H^q}}{\|k\|_{A^q}} \right\}^{q/p},$$

where C_p is a constant depending on p, which may be taken to be the norm of the Bergman projection on A^p (see [5], Theorem 4.2). In [6], it is proved that the converse to Ryabykh's theorem holds when p is an even integer. In fact, Theorem 4.3 in the above reference says that the following holds: Suppose p is an even integer and let q be its conjugate exponent. Let $F \in A^p$ with $\|F\|_{A^p} = 1$, and let k be an integral kernel such that F is the extremal function for the functional corresponding to k. (It is known that k is unique up to a positive scalar multiple, see [5]). If $F \in H^{p_1}$ for some p_1 with $p-1 < p_1 < \infty$, then $k \in H^{q_1}$ for $q_1 = p_1/(p-1)$, and

$$\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \leq C \|F\|_{H^{p_1}}^{p-1},$$

where C is a constant depending only on p and p_1. (The statement in the reference is only for $p_1 \geq p$, but the proof works for all $p_1 > p-1$.) Since by Theorem 2.2 in [7] the function k is a constant multiple of $\mathcal{P}(|F|^{p-2}F)$, this theorem implies the following result: Suppose that $1 < q_1 < \infty$ and that p is an even integer. If g has the form $g = |f|^{p-2}f$ for some analytic function f, and if g has bounded M_{q_1} integral means, then $\mathcal{P}(g) \in H^{q_1}$, where $\mathcal{P}$ is the Bergman projection.

This observation led us to this study of integral means of Bergman projections. We are not able to prove that the above result about the Bergman projection of functions of the form $|F|^{p-2}F$ holds when p is not an even integer. However, as a result of our investigations, we are able to prove that if $1 \leq p-1 \leq p_1 < \infty$ and $0 < \alpha < 1$ and the extremal function F is in H^{p_1} and has boundary values in the Lebesgue-Lipschitz space $\Lambda^{p_1}_{\alpha}$, then the integral kernel $k \in H^{q_1}$ and has boundary values in $\Lambda^{q_1}_{\alpha}$, where $q_1 = p_1/(p-1)$.

1 Hypergeometric Functions and Two Lemmas

We first discuss hypergeometric functions, since we will use them in some of our proofs. The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined for $|z|<1$ and c not a non-positive integer by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n $$

(see [1], eq. 15.2.1), where $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. Note that $(a)_0 = 1$. A hypergeometric function may be analytically continued to a single valued analytic function on $\mathbb{C}$ minus the part of the real axis from 1 to ∞. This analytic continuation is called the principal branch of the hypergeometric function.

For $\operatorname{Re} c > \operatorname{Re} b > 0$, we have that

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx \qquad (1) $$

(see [1], eq. 15.6.1). Also, for $|\arg(1-z)| < \pi$ we have

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z) \qquad (2) $$

(see [1], eq. 15.8.1). Kummer's quadratic transformation states that for $|z| < 1$ we have

$$ {}_2F_1(a,b;2b;4z/(1+z)^2) = (1+z)^{2a}{}_2F_1(a,a+\frac{1}{2}-b;b+\frac{1}{2};z^2) \qquad (3) $$

(see [1], eq. 15.8.21). If $\operatorname{Re} c > \operatorname{Re}(a+b)$ then the power series defining ${}_2F_1$ converges absolutely on the $|z|=1$ (see [1], 15.2.(i)), and thus uniformly on $|z| \le 1$. In addition, for $\operatorname{Re} c > \operatorname{Re}(a+b)$ the value of the hypergeometric function at 1 (that is, the sum of the hypergeometric series for $z=1$) is given by

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (4) $$

(see [1], eq. 15.4.20).

The following lemma is well known, although we do not know if anyone has found the sharp constant before (see e.g. [8], Theorem 1.7).

Lemma 1. *Let $p > 1$ and $0 < r < 1$. Then*

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^p} d\theta &= (1-r^2)^{1-p}{}_2F_1\left(1-\frac{p}{2}, 1-\frac{p}{2}; 1; r^2\right) \\ &\le \frac{\Gamma(p-1)}{\Gamma(p/2)^2}(1-r^2)^{1-p}.\end{aligned}$$

Furthermore, the bound is sharp.

In the case $p = 2$, this says that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^2} d\theta = (1-r^2)^{-1}.$$

Proof. The integral in question is equal to

$$\frac{1}{\pi}\int_0^{\pi} (1 - 2r\cos\theta + r^2)^{-p/2}\, d\theta.$$

Making the substitution $x = (\cos\theta + 1)/2$, we see that the integral is equal to

$$\begin{aligned}&\frac{1}{\pi}(1+r)^{-p}\int_0^1 \left(1 - \frac{4r}{(1+r)^2}x\right)^{-p/2} x^{-1/2}(1-x)^{-1/2} dx \\ &= (1+r)^{-p}{}_2F_1(p/2, 1/2; 1; 4r/(1+r)^2)\end{aligned}$$

by equation (1). Now using equation (3), we see this is equal to

$${}_2F_1(p/2, p/2; 1; r^2).$$

Equation (2) shows this is equal to

$$(1-r^2)^{1-p}{}_2F_1(1-(p/2), 1-(p/2); 1; r^2).$$

The bound now follows from equation (4), since the series representation shows that ${}_2F_1(1-(p/2), 1-(p/2); 1; r^2)$ increases from $r = 0$ to $r = 1$. The remark about $p = 2$ is true because ${}_2F_1(1, 1; 1; x) = (1-x)^{-1}$. □

The following lemma is likely well known, at least without the sharp constant, although we do not know of a specific place where it appears in the literature.

Lemma 2. *Suppose that $s < 1$ and $m + s > 1$ and that $k > -1$. Let $0 \le x < 1$. Then*

$$\int_0^1 \frac{(1-y)^{-s}}{(1-xy)^m} y^k \, dy \le C_1(s,m,k)(1-x)^{1-s-m}$$

where $C_1(s,m,k) < \infty$ is defined by

$$C_1(s,m,k) = \frac{\Gamma(k+1)\Gamma(1-s)}{\Gamma(2+k-s)} \max_{0 \le x \le 1} {}_2F_1(2+k-s-m, 1-s; 2+k-s; x).$$

Furthermore, the bound is sharp.

Proof. By (1), we have

$$\int_0^1 \frac{(1-y)^{-s}}{(1-xy)^m} y^k \, dy = \frac{\Gamma(k+1)\Gamma(1-s)}{\Gamma(2+k-s)} {}_2F_1(m, k+1; 2+k-s; x)$$

Now (2) gives that this is equal to

$$\frac{\Gamma(k+1)\Gamma(1-s)}{\Gamma(2+k-s)} (1-x)^{1-s-m} {}_2F_1(2+k-s-m, 1-s; 2+k-s; x).$$

Now since $s + m - 1 > 0$, the function ${}_2F_1(2+k-s-m, 1-s; 2+k-s; z)$ converges uniformly on $|z| \le 1$, and so ${}_2F_1(2+k-s-m, 1-s; 2+k-s; x)$ is bounded for $|x| \le 1$. Thus, the above displayed expression is less than or equal to $C_1(s,m,k)(1-x)^{1-s-m}$. □

Note that if $2 + k > s + m$ and $2 + k > s$, then the hypergeometric function ${}_2F_1(2+k-s-m, 1-s; 2+k-s; x)$ is increasing on $[0,1)$, and so the maximum in the bound occurs at $x = 1$. By (4), $C_1(s,m,k)$ becomes

$$\frac{\Gamma(2+k-s)\Gamma(s+m-1)\Gamma(k+1)\Gamma(1-s)}{\Gamma(1+k)\Gamma(m)\Gamma(2+k-s)} = \frac{\Gamma(s+m-1)\Gamma(1-s)}{\Gamma(m)}.$$

2 Bounds on Integral Means of Bergman Projections

As discussed above, we make the following definition.

Definition 1. *Let $f \in L^p(\mathbb{D})$ for $0 < p \le \infty$. Define*

$$M_p(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}$$

for $0 < p < \infty$ and

$$M_p(r,f) = \operatorname*{ess\,sup}_{0 \le \theta < 2\pi} |f(re^{i\theta})|$$

for $p = \infty$.

Note that for $f \in L^p(\mathbb{D})$, the integral means $M_p(r,f)$ are defined for almost every r such that $0 < r < 1$, and in fact the function $M_p(\cdot, f)$ is in $L^p(r\,dr)$ on $[0,1)$. For $0 < p < \infty$ this follows immediately from Fubini's theorem (see [12], Theorem 7.12), and for $p = \infty$ it may be proved either directly with the aid of Fubini's theorem, or by noting that $M_\infty(r,f) = \lim_{p\to\infty} M_p(r,f)$.

We next define an auxiliary operator which we will use to help bound the Bergman projection.

Definition 2. *Let $f \in L^1(\partial\mathbb{D})$. Define*

$$\mathcal{P}_r^{(n)}(f)(\theta) = \frac{(n+1)!}{2\pi} \int_0^{2\pi} \frac{f(e^{i\phi}) r^n e^{-in\phi}}{(1 - re^{i(\theta-\phi)})^{2+n}} d\phi.$$

We now have the following theorem, which gives a bound on the L^p norm of $\mathcal{P}_r^{(n)}(f)$.

Theorem 3. *Let $1 \le p \le \infty$, and let k be an integer such that $0 \le k \le n$. Assume that f is in the Sobolev space $W^{k,p}(\partial\mathbb{D})$. Then*

$$\|\mathcal{P}_r^{(n)}(f)\|_p \le \frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2} r^{n-k}(1-r^2)^{k-n-1} \left\| \frac{d^k}{d\theta^k} \left[f(e^{i\theta}) e^{-in\theta} \right] \right\|_p,$$

where $\|\cdot\|_p$ denotes the $L^p(\partial\mathbb{D})$ norm.

Proof. First assume that $p < \infty$. Performing integration by parts k times gives

$$\begin{aligned} \mathcal{P}_r^{(n)}(f)(\theta) &= \frac{(n+1)!}{2\pi} \int_0^{2\pi} \frac{f(e^{i\phi}) r^n e^{-in\phi}}{(1 - re^{i(\theta-\phi)})^{2+n}} d\phi \\ &= r^{n-k} e^{-ik\theta} \frac{(n-k+1)!}{2\pi i^k} \int_0^{2\pi} \frac{\frac{d^k}{d\theta^k}[f(e^{i\phi}) e^{-in\phi}]}{(1 - re^{i(\theta-\phi)})^{2+n-k}} d\phi. \end{aligned}$$

This is legitimate since f is in $W^{k,p}$, and thus all its derivatives except possibly the k^{th} are continuous. We have also used the fact that both $f(e^{i\phi})$ and $(1-re^{i(\theta-\phi)})^{-1}$ are periodic in ϕ with period 2π.

The above displayed equation, Lemma 1 and Hölder's inequality immediately gives the case $p=\infty$. If $p<\infty$, let $m=n+2-k$, and let $g(e^{i\theta})=\frac{d^k}{d\phi^k}[f(e^{i\phi})e^{-in\phi}]$. Note that

$$(r^{n-k}(n-k+1)!)^{-p}\|\mathcal{P}_r^{(n)}(f)\|_p^p \le \int_0^{2\pi}\left|\int_0^{2\pi}\frac{|g(e^{i\phi})|}{|1-re^{i(\theta-\phi)}|^m}\frac{d\phi}{2\pi}\right|^p\frac{d\theta}{2\pi}.$$

But the right hand side of the above inequality equals

$$\int_0^{2\pi}\left|\int_0^{2\pi}\frac{|g(e^{i\phi})|}{|1-re^{i(\theta-\phi)}|^{m/p}}\frac{1}{|1-re^{i(\theta-\phi)}|^{m/q}}\frac{d\phi}{2\pi}\right|^p\frac{d\theta}{2\pi},$$

where q is the conjugate exponent to p. By Hölder's inequality, this is less than or equal to

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|g(e^{i\phi})|^p}{|1-re^{i(\theta-\phi)}|^m}\frac{d\phi}{2\pi}\left|\int_0^{2\pi}\frac{1}{|1-re^{i(\theta-\phi)}|^m}\frac{d\phi}{2\pi}\right|^{p/q}\frac{d\theta}{2\pi}.$$

And by Lemma 1, this is at most

$$\left(\frac{\Gamma(m-1)}{\Gamma(m/2)^2}\right)^{p-1}\int_0^{2\pi}\int_0^{2\pi}\frac{|g(e^{i\phi})|^p}{|1-re^{i(\theta-\phi)}|^m}\frac{d\phi}{2\pi}(1-r^2)^{(1-m)(p-1)}\frac{d\theta}{2\pi},$$

where we have used the fact that $p/q=p-1$. Now Tonelli's theorem shows that this equals

$$\begin{aligned}&\left(\frac{\Gamma(m-1)}{\Gamma(m/2)^2}\right)^{p-1}(1-r^2)^{(1-m)(p-1)}\int_0^{2\pi}|g(e^{i\phi})|^p\int_0^{2\pi}\frac{1}{|1-re^{i(\theta-\phi)}|^m}\frac{d\theta}{2\pi}\frac{d\phi}{2\pi}\\ \le&\left(\frac{\Gamma(m-1)}{\Gamma(m/2)^2}\right)^{p-1}\frac{\Gamma(m-1)}{\Gamma(m/2)^2}(1-r^2)^{(1-m)(p-1)}(1-r^2)^{1-m}\int_0^{2\pi}|g(e^{i\phi})|^p\frac{d\phi}{2\pi}\\ =&\left(\frac{\Gamma(m-1)}{\Gamma(m/2)^2}\right)^{p}(1-r^2)^{(1-m)p}\|g\|_p^p,\end{aligned}$$

where we have again applied Lemma 1. This proves the result for $p<\infty$. (Note that in the case $p=1$, the above proof still works and really only involves Lemma 1 and Tonelli's theorem, but not Hölder's inequality.)

□

For $f \in L^1(\mathbb{D})$, recall that the Bergman projection of f is defined by

$$\mathcal{P}f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w),$$

and thus

$$\frac{d^n}{dz^n}(\mathcal{P}f)(z) = \frac{(n+1)!}{\pi}\int_{\mathbb{D}} \frac{f(w)\overline{w}^n}{(1-\overline{w}z)^{2+n}} dA(w). \tag{5}$$

Therefore, if $z = re^{i\theta}$, we have that

$$\begin{aligned}\frac{d^n}{dz^n}(\mathcal{P}f)(z) &= \frac{(n+1)!}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{f(\rho e^{i\phi})\rho^n e^{-in\phi}}{(1-r\rho e^{i(\theta-\phi)})^{2+n}}\, d\phi\, d\rho \\ &= \frac{(n+1)!}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{f_\rho(e^{i\phi})\rho^n e^{-in\phi}}{(1-r\rho e^{i(\theta-\phi)})^{2+n}}\, d\phi\, d\rho, \\ &= 2\int_0^1 \rho r^{-n} \mathcal{P}_{r\rho}^{(n)} f_\rho(e^{i\theta})\, d\rho,\end{aligned}$$

where $f_\rho(e^{i\theta}) = f(\rho e^{i\theta})$.

Theorem 4. *Let $1 \le p \le \infty$, and let k and n be integers such that $0 \le k \le n$. Suppose that $f \in L^1(\mathbb{D})$, and that the restriction of f to almost every circle of radius less than 1 centered at the origin is in $W^{k,p}$. Then the following inequality holds:*

$$\begin{aligned}M_p\left(r, \frac{d^n}{dz^n}(\mathcal{P}f(z))\right) \le 2\frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2} \quad \times \\ r^{-k}\int_0^1 \rho^{n+1-k} M_p\left(\frac{d^k}{d\theta^k}(e^{-in\theta}f), \rho\right)(1-r^2\rho^2)^{k-n-1} d\rho.\end{aligned}$$

Since

$$\left|\frac{d^k}{d\theta^k}(e^{-in\theta}f),\right| \le \sum_{j=0}^{k}\binom{k}{j} n^{k-j}\left|\frac{d^j}{d\theta^j}f\right|, \tag{6}$$

it is not hard to use the above theorem to bound $M_p(r, (\mathcal{P}f)^{(n)})$ strictly in terms of the integral means of the first k derivatives of f in the θ direction.

Proof. Again, first assume that $p < \infty$. We have that

$$M_p\left(r, \frac{d^n}{dz^n}(\mathcal{P}f(z))\right) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|2\int_0^1 \rho r^{-n}\mathcal{P}_{r\rho}^{(n)} f_\rho(e^{i\theta}) d\rho\right|^p d\theta\right)^{1/p}.$$

By Minkowski's inequality, this is less than or equal to

$$2\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \rho^p r^{-pn} |\mathcal{P}^{(n)}_{r\rho} f_\rho(e^{i\theta})|^p d\theta\right)^{1/p} d\rho.$$

By Theorem 3, this is less than or equal to

$$2\frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2} r^{-k} \int_0^1 \rho^{n+1-k} \left\| \frac{d^k}{d\theta^k}(e^{-in\theta} f_\rho)\right\|_p (1-r^2\rho^2)^{k-n-1} d\rho$$

which equals

$$2\frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2} r^{-k} \int_0^1 \rho^{n+1-k} M_p\left(\frac{d^k}{d\theta^k}(e^{-in\theta} f), \rho\right) (1-r^2\rho^2)^{k-n-1} d\rho.$$

The proof is slightly easier in the case $p=\infty$, as we do not need Minkowski's inequality. Alternately, to see that the theorem still holds for $p=\infty$, we can take the limit in the bound as $p=\infty$, using the monotone convergence theorem and the fact that $M_p(r,f)$ increases with p. □

We now discuss Lipschitz and Lebesgue-Lipschitz classes, since they are relevant to some corollaries which we are about to prove. A function f is said to be Lipschitz of order α for $0<\alpha\le 1$ if there is some constant A such that $|f(x)-f(y)| \le A|x-y|^\alpha$ for all x and y in its domain. The class of all such functions is denoted by Λ_α. For a function f defined on the unit circle, we define its integral modulus of continuity of order p for $p<\infty$ by

$$\omega_p(t,f) = \sup_{0<h\le t}\left[\frac{1}{2\pi}\int_0^{2\pi} |f(x+h)-f(x)|^p\right]^{1/p}.$$

If $\omega_p(t,f) = O(t^\alpha)$ for some α such that $0<\alpha\le 1$, we say that f belongs to the Lebesgue-Lipschitz class Λ^p_α. For $p=\infty$, we define $\Lambda^\infty_\alpha = \Lambda_\alpha$.

We will need Theorem 5.4 in [3], which states that an analytic function is in H^p for $1\le p<\infty$ and has boundary values in Λ^p_α if and only if the integral means of its derivative satisfy $M_p(r,f') = O((1-r)^{-1+\alpha})$. We will also use Theorem 5.1 from the same reference, which says that an analytic function is in H^∞ and has boundary values in $\Lambda_\alpha = \Lambda^\infty_\alpha$ if and only if the integral means of its derivative satisfy $M_\infty(r,f') = O((1-r)^{-1+\alpha})$. (As stated, the theorem has the function being continuous in $\overline{\mathbb{D}}$ in place of its being in H^∞, but any analytic function in H^∞ with Lipschitz (or even continuous) boundary values is continuous in $\overline{\mathbb{D}}$.)

Corollary 5. *Let* $1 \le p \le \infty$ *and let* $0 < \alpha < 1$. *Suppose that* f *is measurable in* $\mathbb{D}$ *and that the restriction of* f *to almost every circle of radius less than* 1 *centered at the origin is in* $W^{1,p}$. *If* $M_p\left(\frac{d}{d\theta}f, r\right) = O((1-r)^{-1+\alpha})$ *and* $M_p(f,r) = O((1-r)^{-1+\alpha})$, *then* $\mathcal{P}(f) \in H^p$, *and in fact the boundary values of* $\mathcal{P}(f)$ *are in the Lebesgue-Lipschitz space* Λ^p_α .

Proof. Note that the assumptions imply that $M_p(r, \frac{d}{d\theta}(e^{-i\theta}f)) = O((1-r)^{-1+\alpha})$, and that $f \in L^1(\mathbb{D})$. By the theorem,

$$M_p(r, \mathcal{P}(f)') \le C\int_0^1 (1-r)^{-1+\alpha}(1-r^2\rho^2)^{-1}d\rho \le C\int_0^1 (1-r)^{-1+\alpha}(1-r\rho)^{-1}d\rho$$

for r near enough to 1, where C is a constant. By Lemma 2, the above expression is less than or equal to $C(1-r)^{-1+\alpha}$ for r near enough to 1, where C is another constant. But this implies that $f \in H^p$ and has boundary values in Λ^p_α. □

We now state a corollary related to our original motivation for studying this problem. If we are given an $f \in A^p$ with unit norm, where $1 < p < \infty$ and p has conjugate exponent q, then there is a function $k \in A^q$ (unique up to a positive scalar multiple) such that f solves the extremal problem of maximizing $\operatorname{Re}\int_{\mathbb{D}} g\overline{k}\, d\sigma$ among all functions g of unit A^p norm. The broad question that first motivated our study was: if we know that f has certain regularity, can we say anything about regularity properties for k? The next corollary is an example of this.

Corollary 6. *Let* $2 \le p < \infty$, *let* $p-1 \le s \le \infty$, *let* q *be the conjugate exponent to* p, *and let* $0 < \alpha < 1$. *Let* f *be analytic and suppose that* $f \in H^s$ *with boundary values in* Λ^s_α, *and that* $\|f\|_{A^p} = 1$. *Let* k *a function in* A^q *such that* f *solves the extremal problem of finding a function* g *of unit* A^p *norm maximizing* $\operatorname{Re}\int_{\mathbb{D}} g\overline{k}\, d\sigma$. *Then* $k \in H^{s/(p-1)}$ *and the boundary values of* k *are in* $\Lambda^{s/(p-1)}_\alpha$.

Proof. By the above mentioned Theorem 5.4 from [3], we have that $M_s(r, f') = O((1-r)^{-1+\alpha})$. Now, if we write $f = u + iv$, we have

$$\frac{\partial}{\partial\theta}(|f|^{p-2}f) = (p-2)|f|^{p-4}[uu_\theta + vv_\theta]f + |f|^{p-2}f_\theta.$$

The absolute value of the above expression is bounded by $(2p-3)|f|^{p-2}|f_z|$, where we have used the fact that $f_\theta = izf'$, and the inequality $ab + cd \le$

$(a+c)(b+d) \leq 2\sqrt{a^2+c^2}\sqrt{b^2+d^2}$, which holds when $a, b, c, d \geq 0$. Thus we have

$$M_{s/(p-1)}\left(r, \frac{\partial}{\partial\theta}(|f|^{p-2}f)\right) \leq (2p-3)\|f\|_{H^s}^{p-2} M_s(r, f') \leq C(1-r)^{-1+\alpha},$$

where in the first inequality we have used Hölder's inequality, and in the second we have used the hypothesis about the growth of the integral means of f'. Also, it is clear that $M_{s/(p-1)}(r, |f|^{p-2}f)$ is bounded. By the previous lemma, this implies that $\mathcal{P}(|f|^{p-2}f) \in H^{s/(p-1)}$ and that $\mathcal{P}(|f|^{p-2}f)$ has boundary values in $\Lambda_\alpha^{s/(p-1)}$. But since $\mathcal{P}(|f|^{p-2}f)$ is a constant multiple of k, the corollary holds. □

This corollary is similar to Theorem 4.3 in [6], which is proved by very different methods. That theorem is only proved for p an even integer. It requires us assume that $f \in H^s$ for $p-1 < s < \infty$, and yields that $k \in H^{s/(p-1)}$. Whether Theorem 4.3 from [6] holds when p is not an even integer is still an open question.

3 Bounds on Sobolev norms of Bergman Projections

We now illustrate how our previous results can be used to bound certain weighted L^p norms of derivatives of Bergman projections by other weighted L^p norms of θ derivatives of the original function. We will need the following lemma.

Lemma 7. *Suppose $1 < p < \infty$ and that $j, k > -1$ and $m > 0$ and $u < 1$, and that $u > 1 - mp$. Set $w = u + (m-1)p$. For a measurable function f define*

$$g(x) = \int_0^1 \frac{|f(y)|}{(1-xy)^m} y^k \, dy,$$

where we allow $g(x)$ to be infinite in places. Then

$$\|g\|_{L^p(x^j(1-x)^{-u}\,dx)} \leq C_2 \|f\|_{L^p(x^k(1-x)^{-w}\,dx)},$$

where L^p spaces in the bound are on the interval $[0,1]$, and where

$$C_2 = C_2(p, m, k, j, u) = \inf_{\substack{0<b<m \\ a \text{ satisfies all of } (8)}} C_1(aq, (m-b)q, k)^{1/q} C_1(ap + (m-b)p + u - (p/q), bp, j)^{1/p}.$$

Proof. Let q be the conjugate exponent to p. Choose b so that $0 < b < m$. First note that the above conditions imply that

$$1 - m - \frac{u}{p} < \frac{1}{q} \tag{7a}$$

$$1 - \frac{u}{p} - m < 1 - \frac{u}{p} - (m - b) \tag{7b}$$

$$\frac{1}{q} - (m - b) < \frac{1}{q} \tag{7c}$$

$$\frac{1}{q} - (m - b) < 1 - \frac{u}{p} - (m - b) \tag{7d}$$

Thus we can find a number a satisfying

$$1 - m - \frac{u}{p} < a \tag{8a}$$

$$\frac{1}{q} - (m - b) < a \tag{8b}$$

$$a < \frac{1}{q} \tag{8c}$$

$$a < 1 - \frac{u}{p} - m + b \tag{8d}$$

We may assume without loss of generality that $f \geq 0$, since if the inequality holds for $|f|$ it holds for f. Now

$$\begin{aligned} &\int_0^1 \frac{f(y)}{(1-xy)^m} y^k \, dy \\ &= \int_0^1 \frac{f(y)(1-y)^a}{(1-xy)^b} \frac{(1-y)^{-a}}{(1-xy)^{m-b}} y^k dy \\ &\leq \left[\int_0^1 \frac{|f(y)|^p (1-y)^{ap}}{(1-xy)^{bp}} y^k dy \right]^{1/p} \times \left[\int_0^1 \frac{(1-y)^{-aq}}{(1-xy)^{(m-b)q}} y^k dy \right]^{1/q}, \end{aligned}$$

by Hölder's inequality. But by Lemma 2, the above expression is less than or equal to

$$C_{1,1}^{1/q} (1-x)^{(1/q)-a-(m-b)} \left[\int_0^1 \frac{|f(y)|^p (1-y)^{ap}}{(1-xy)^{bp}} y^k dy \right]^{1/p},$$

where $C_{1,1} = C_1(aq, (m-b)q, k)$. This is valid because $aq + (m-b)q > 1$ and $aq < 1$, which follow from inequalities (8b) and (8c). So then

$$\begin{aligned}
&\|g\|^p_{L^p(x^j(1-x)^{-u}\,dx)} \\
&= \int_0^1 \left| \int_0^1 \frac{f(y)}{(1-xy)^m} y^k\,dy \right|^p (1-x)^{-u} x^j\,dx \\
&\le C_{1,1}^{p/q} \int_0^1 (1-x)^{(p/q)-ap-(m-b)p} \int_0^1 \frac{|f(y)|^p (1-y)^{ap}}{(1-xy)^{bp}} y^k\,dy\,(1-x)^{-u} x^j\,dx \\
&= C_{1,1}^{p/q} \int_0^1 |f(y)|^p (1-y)^{ap} \int_0^1 \frac{(1-x)^{(p/q)-ap-(m-b)p-u}}{(1-xy)^{bp}}\, x^j\,dx\, y^k\,dy,
\end{aligned}$$

by Tonelli's theorem for nonnegative functions. Applying the previous lemma again we see that this is less than or equal to

$$\begin{aligned}
&C_{1,1}^{p/q} C_{1,2} \int_0^1 |f(y)|^p (1-y)^{ap} (1-y)^{1+(p/q)-ap-(m-b)p-u-bp} y^k\,dy \\
&= C_{1,1}^{p/q} C_{1,2} \int_0^1 |f(y)|^p (1-y)^{-w} y^k\,dy \\
&= C_{1,1}^{p/q} C_{1,2} \|f\|^p_{L^p(x^k(1-x)^{-w}\,dx)}
\end{aligned}$$

where $C_{1,2} = C_1(ap + (m-b)p + u - (p/q), bp, j)$ This works because $u + (m-b)p + ap - \frac{p}{q} < 1$ and $u + mp + ap - \frac{p}{q} > 1$, which follow from inequalities (8d) and (8a), respectively. □

Note that the proof works even if g is infinite in places, since Hölder's inequality holds even if the left or right sides are infinite, and Tonelli's theorem holds even if some of the integrals involved are infinite.

One important case is when $j = k = m = 1$ and $u = 0$. In this case we can choose $a = 1/(pq)$ and $b = 1/p$, and then we see that $C_2(p, 1, 1, 1, 0) \le C_1(1/p, 1, 1)^{1/q} C_1(1/q, 1, 1)^{1/p}$. (We have tried to find a choice of a and b yielding a better bound on C_2, but were not able). But by the remarks after Lemma 2, this is equal to

$$\left[\frac{\Gamma(1/p)\Gamma(1/q)}{\Gamma(1)}\right]^{1/q} \left[\frac{\Gamma(1/q)\Gamma(1/p)}{\Gamma(1)}\right]^{1/p} = \Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)}$$

by the reflection formula for the Γ function.

It is interesting to note that the bounds in the following theorem do not depend on p.

Theorem 8. *Let $1 \le p \le \infty$ and $1 < s < \infty$. Suppose that $0 \le k \le n$, where n and k are integers. Also suppose that $j-k > -1$ and $1-(n+1-k)s < u < 1$ and set $w = u + (n-k)s$. Also suppose that the restriction of f to almost every circle of radius less than 1 centered at the origin is in $W^{k,p}$, and that f is in $L^1(\mathbb{D})$. Then*

$$\left\{ \int_0^1 [M_p(r, (\mathcal{P}f)^{(n)})]^s \, (1-r)^{-u} r^j \, dr \right\}^{1/s}$$

$$\le C_3(s, n-k, j-k, u) \left\{ \int_0^1 \left[M_p \left(\frac{d^k}{d\theta^k}(e^{-in\theta} f), r \right) \right]^s (1-r)^{-w} r^{n-k+1} \, dr \right\}^{1/s}$$

where

$$C_3(s, n-k, j-k, u) = \frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2} C_2(s, n-k+1, n-k+1, j-k, u).$$

Proof. Define

$$g(r) = \int_0^1 \rho^{n+1-k} M_p \left(\frac{d^k}{d\theta^k}(e^{-in\theta} f), \rho \right) (1-\rho r)^{k-n-1} d\rho.$$

Then by Theorem 4 and the fact that $(1-\rho^2 r^2)^{k-n-1} \le (1-\rho r)^{k-n-1}$ we have

$$M_p(r, (\mathcal{P}f)^{(n)}) \le C r^{-k} g(r),$$

where $C = 2\frac{\Gamma(n+1-k)\Gamma(n+2-k)}{\Gamma((n+2-k)/2)^2}$. But by Lemma 7,

$$\left(\int_0^1 |g(r)|^s \, (1-r)^{-u} \, r^{j-k} dr \right)^{1/s}$$

$$\le C_2(s, n-k+1, n-k+1, j-k, u) \quad \times$$

$$\left[\int_0^1 \left[M_p \left(\frac{d^k}{d\theta^k}(e^{-in\theta} f), r \right) \right]^s (1-r)^{-w} r^{n-k+1} \, dr \right]^{1/s}.$$

□

By using Equation (6), it is not hard to modify the bound in the theorem so that it only involves the integral means of the first k derivatives of f in the θ direction.

Note that if we take $1 < s < \infty$ and $n = k$ and $j = 1+n$ and $u = 0$, then by the remarks after Lemma 7 we see that $C_3(s, 0, 1, 0) \le 2\pi/\sin(\pi s)$. If we also take $p = s$ and note that $r \, dr \, d\theta = dA$, we have the following corollary.

Corollary 9. *For $1 < p < \infty$ and $n \geq 0$, if $f \in L^1(\mathbb{D})$ and f is in $W^{n,p}$ when restricted to almost every circle of radius less than 1 centered at the origin, we have*

$$\|\mathcal{P}(f)^{(n)}\|_{L^p(r^n dA)} \leq 2\frac{\pi}{\sin(\pi/p)} \left\| \frac{d^n}{d\theta^n}(e^{-in\theta} f) \right\|_{L^p(dA)}.$$

Here is another corollary, which follows from taking $p = s$, replacing n with $n + k$ where $n, k \geq 0$, and letting $j = 1 + k$, and $u = 0$.

Corollary 10. *For $1 < p < \infty$ and integers $n, k \geq 0$, if $f \in L^1(\mathbb{D})$ and f is in $W^{k,p}$ when restricted to almost every circle of radius less than 1 centered at the origin we have*

$$\|\mathcal{P}(f)^{(n+k)}\|_{L^p(r^k dA)} \leq C_3(p, n, 1, 0) \left\| \frac{d^k}{d\theta^k}(e^{-i(n+k)\theta} f) \right\|_{L^p(r^n(1-r)^{-np}\, dA)}.$$

If $k = 0$, the right hand side above simplifies to

$$C_3(p, n, 1, 0)\|f\|_{L^p(r^n(1-r)^{-np}\, dA)}.$$

Now, if we take $b = 1/p$ and $a = -n + 1/(pq)$ in the definition of C_2, we see that

$$\begin{aligned}
&C_3(p, n, 1, 0) \\
&\leq 2\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(1 + (n/2))} \times \\
&\quad C_1(aq, (n + 1 - b)q, n + 1)^{1/q} C_1(ap + (n + 1 - b)p - (p - 1), bp, 1)^{1/p} \\
&= 2\frac{\Gamma(n+1)\Gamma(n+2)\Gamma(1/p)^{1/q}\Gamma(nq + (1/q))^{1/q}\Gamma(1/q)^{1/p}\Gamma(1/p)^{1/p}}{\Gamma(1 + (n/2))\Gamma(nq + 1)^{1/q}\Gamma(1)^{1/p}} \\
&= 2\frac{\Gamma(n+1)\Gamma(n+2)\Gamma(1/p)\Gamma(1/q)^{1/p}\Gamma(nq + (1/q))^{1/q}}{\Gamma(1 + (n/2))\Gamma(nq + 1)^{1/q}}.
\end{aligned}$$

4 A Counterexample

We now give an example of a function f such that $M_2(r, f)$ is bounded but $\mathcal{P}f$ is not in H^2. In fact, the function in our example can be chosen so that $M_\infty(r, f) \to 0$ as $r \to 1^-$.

We first derive some general formulas for the Bergman projection of a function. Suppose that $f \in L^2(\mathbb{D})$. Note that for almost every r in $[0,1]$, f restricted to the circle of radius r has a Fourier series since it is in $L^2([0,2\pi))$ for almost every r. Thus we can write

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta},$$

where for a.e. r convergence holds in $L^2(0,2\pi)$. Here

$$a_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta.$$

Note that the functions $a_n(r)$ are measurable by Fubini's theorem. Also, by Fubini's theorem

$$\int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{2\pi} = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta\, r\,dr.$$

But since, for almost every fixed r, the Fourier series in θ of $f(re^{i\theta})$ converges in $L^2(0,2\pi)$, we have

$$\int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{2\pi} = \frac{1}{\pi}\int_0^1 \sum_{n=0}^{\infty} |a_n(r)|^2\, r\,dr. \tag{9}$$

We now prove the following lemma relating what we have said to calculating the Bergman projection of f.

Lemma 11. *Suppose that $f \in L^2(\mathbb{D})$. Then we can write*

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta}$$

for a.e. r, where for a.e. r, convergence holds in $L^2(0,2\pi)$. Also, the Bergman projection of f is given by

$$(\mathcal{P}f)(z) = \frac{1}{\pi}\sum_{n=0}^{\infty}\left[\int_0^1 (n+1)a_n(r)r^{n+1}dr\right] z^n.$$

Proof. Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Note that

$$\begin{aligned}
(\mathcal{P}f)(z) &= \int_{\mathbb{D}} \sum_{n=0}^{\infty} (n+1) z^n \overline{w}^n f(w)\, d\sigma(w) \\
&= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (n+1) a_m(\rho) e^{im\phi} r^n \rho^{n+1} e^{in\theta} e^{-in\phi}\, d\phi\, d\rho
\end{aligned}$$

by Fubini's theorem. For fixed z and ρ, the sum $\sum_{n=0}^{\infty}(n+1)z^n\overline{w}^n$ converges uniformly on $[0, 2\pi]$, and thus for fixed z and almost every fixed ρ, the sum $\sum_{n=0}^{\infty}(n+1)z^n\overline{w}^n f(w)$ converges in $L^2([0, 2\pi])$. Also, for almost every fixed ρ the sum $\sum_{m=-\infty}^{\infty} a_m(\rho)e^{im\phi}$ converges in $L^2(0, 2\pi)$. Thus, we can move the integral over ϕ inside the two summations to see that

$$(\mathcal{P}f)(z) = \frac{1}{\pi} \int_0^1 \sum_{n=0}^{\infty} (n+1) a_n(\rho) \rho^{n+1} r^n e^{in\theta}\, d\rho.$$

Now, we wish to apply the dominated convergence theorem to move the sum outside the integral. To see that we can do this, note that for each ρ and each $N \geq 0$ we have by the Cauchy-Schwarz inequality that

$$\left| \sum_{n=0}^{N} (n+1) a_n(\rho) \rho^n r^n e^{in\theta} \right| \leq \left(\sum_{n=0}^{\infty} |a_n(\rho)|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} (n+1)^2 r^{2n} \rho^{2n} \right)^{1/2}.$$

But the second sum can be bounded by $(1 + r^2)/(1 - r^2)^3$ independently of ρ and the first sum is integrable in with respect to the measure $\rho\, d\rho$ by equation (9). Thus we may apply the dominated convergence theorem to see that

$$(\mathcal{P}f)(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \left[\int_0^1 (n+1) a_n(\rho) \rho^{n+1}\, d\rho \right] z^n.$$

□

Of course, this theorem shows that the formula for the Bergman projection is valid if $f \in L^p$ for $p > 2$, since L^p is then a subset of L^2. The formula also holds for any $f \in L^p$ for $p > 1$. This can be shown by using the fact that the Fourier series of an L^p function converges to that function in L^p for $1 < p < \infty$, by using Hölder's inequality instead of the Cauchy-Schwarz inequality, and by using Fubini's theorem and the Hausdorff-Young inequality to show that $\sum_{n=0}^{\infty} |a_n(\rho)|^q$ is integrable with respect to $\rho\, d\rho$. However, we will really only need the formula to hold for bounded functions, since the functions to which we need to apply the theorem will all be bounded.

To construct a function f such that $M_\infty(r, f) \to 0$ as $r \to 1^-$ and $\mathcal{P}f \notin H^2$, we will use the following lemma. Note that the constant $1/4$ in the lemma is not sharp and could be replaced any number strictly between 0 and 1.

Lemma 12. *There is an increasing sequence* $0 = b_0, b_1, b_2, \ldots \to 1$ *and an increasing sequence of non-negative integers* $m_1, m_2, \ldots$ *such that*

$$\int_{b_{n-1}}^{b_n} (m_n + 1) r^{m_n+1}\, dr \geq \frac{1}{4}.$$

Proof. We prove this by induction. Let $b_0 = 0, b_1 = 1/\sqrt{2}$, and $m_1 = 0$. Then we have

$$\int_{b_0}^{b_1} (m_1 + 1) r^{m_1+1}\, dr = \int_0^{1/\sqrt{2}} r\, dr = \frac{1}{4}.$$

Now, suppose we have found an increasing sequence of constants $b_0, \ldots, b_n < 1$ and an increasing sequence of non-negative integers $m_1, \ldots m_n$ satisfying the above condition. Note that for each $k \geq 0$,

$$\int_{b_n}^{1} (k + 1) r^{k+1}\, dr = \frac{k+1}{k+2}(1 - b_n^{k+1}).$$

Now, as $k \to \infty$, this approaches 1, so there is some k such that

$$\int_{b_n}^{1} (k + 1) r^{k+1}\, dr = \frac{k+1}{k+2}(1 - b_n^{k+1}) \geq \frac{1}{2}.$$

We choose m_{n+1} to be the smallest such k. Now, the above inequality implies that there is some constant b such that $b_n < b < 1$ and

$$\int_{b_n}^{b} (k + 1) r^{k+1}\, dr = \frac{1}{4}.$$

We then choose $b_{n+1} = b$. □

We now have the following theorem in which we construct bounded functions whose Bergman projections are not in H^2.

Theorem 13. *Let the b_n be defined as in the previous lemma. Let $\{c_n\}_{n=1}^{\infty}$ be a bounded sequence such that $\sum_n |c_n|^2 = \infty$. Define*

$$a_j(r) = c_j \chi_{[b_{j-1}, b_j)}(r)$$

and

$$f(re^{i\theta}) = \sum_{j=1}^{\infty} a_j(r) e^{im_j\theta}.$$

Then f is bounded but $\mathcal{P}f$ is not in H^2.

Proof. For each r there is exactly one j such that $r \in [b_{j-1}, b_j)$, which implies that $M_\infty(r, f) = c_j$, where j is the number such that $r \in [b_{j-1}, b_j)$. Note that this implies that f is bounded. Thus we have that

$$(\mathcal{P}f)(z) = \frac{1}{\pi} \sum_{n=1}^{\infty} c_n \left[\int_{b_{n-1}}^{b_n} (m_n + 1) r^{m_n+1}\, dr \right] z^{m_n}.$$

But this means that the ${m_n}^{\text{th}}$ term in the Taylor series of $\mathcal{P}f$ is at least $c_n/4$ in absolute value, so that the Taylor coefficients of $\mathcal{P}f$ are not square summable. □

In the theorem, if we choose the sequence $\{c_n\}$ so that it approaches 0 (but is not square summable), then the function f defined in the statement of the theorem will approach 0 uniformly as z approaches the boundary of the disc, but its Bergman projection will not be in H^2. Thus, we have the following corollary.

Corollary 14. *There is a bounded function $f(z)$ in $\mathbb{D}$ that approaches 0 uniformly as $|z| \to 1$, such that $\mathcal{P}f \notin H^2$.*

We note in passing that if we define $a_j(r) = c_j \phi_j(r)$, where ϕ_j is a C^∞ bump function with support in (b_{j-1}, b_j) that is equal to 1 on a sufficiently large part of (b_{j-1}, b_j) then we can even construct f so that it is in C^∞ and approaches 0 uniformly as $|z| \to 1$.

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