Solution of Extremal Problems in Bergman Spaces Using the Bergman Projection

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Abstract

In this paper we discuss the explicit solution of certain extremal problems in Bergman spaces. In order to do this, we develop methods to calculate the Bergman projections of various functions. As a special case, we deal with canonical divisors for certain values of p.

1 Introduction

This paper deals with linear extremal problems in Bergman spaces. The study of extremal problems in Bergman spaces was inspired by extremal problems in Hardy spaces, which have been studied by various authors, notably by Macintyre and Rogosinski (see [18]), Rogosinski and Shapiro (see [20]), and S. Ya. Khavinson (see [14] and [15]).

Bergman space extremal problems have been studied by various authors, for example in [13], [28], [21], [11], and [6]. See also the survey [2]. Regularity results for these problems have been studied in [22], [12], [8], and [9]. However, there are still no general methods for finding solutions to these problems, and few explicit solutions are available. This is in contrast to the situation for Hardy spaces, where a rich theory based on duality and functional analysis allows many extremal problems to be explicitly solved (see the references in the previous paragraph.)

This paper introduces methods for finding explicit solutions to certain extremal problems in Bergman spaces. For example, we solve certain minimal interpolation problems involving finding the smallest norm of a Bergman space function when its value and the value of its first two derivatives are specified at the origin. Similar results to ours are obtained in other works, for example [13], [19], and [24]. As another example, we find the function that maximizes the functional defined by $f \mapsto f^{(n)}(0) + bf(0)$ for certain values of b. The methods are based on theorems developed in the paper about the relation between the Bergman projection and extremal problems, as well as calculations of Bergman projections of various functions. As a special case, we deal with canonical divisors, also known as contractive divisors, for certain A^p spaces.

An analytic function f in the unit disc $\mathbb{D}$ is said to belong to the Bergman space A^p if

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} |f(z)|^p \, d\sigma(z) \right\}^{1/p} < \infty.$$

Here σ denotes normalized area measure, so that $\sigma(\mathbb{D}) = 1$.

For $1 < p < \infty$, the dual of the Bergman space A^p is isomorphic to A^q, where $1/p + 1/q = 1$, and $k \in A^q$ represents the functional defined by

$$\phi(f) = \int_{\mathbb{D}} f(z)\overline{k(z)} \, d\sigma(z).$$

Note that this isomorphism is actually conjugate-linear. It is not an isometry unless $p = 2$, but if the functional $\phi \in (A^p)^*$ is represented by the function $k \in A^q$, then

$$\|\phi\|_{(A^p)^*} \leq \|k\|_{A^q} \leq C_p \|\phi\|_{(A^p)^*} \tag{1.1}$$

where C_p is a constant depending only on p.

In this paper the only Bergman spaces we consider are those with $1 < p < \infty$. The case $p \leq 1$ is more difficult because the proof of Theorem 2.2 fails for $p \leq 1$. This theorem is a key result needed for our method of solving extremal problems. The proof of Theorem 2.2 relies on the boundedness of the Bergman projection on L^p, which fails for $p \leq 1$. It also relies on Hölder's inequality, which fails for $p < 1$.

For a given linear functional $\phi \in (A^p)^*$ such that $\phi \neq 0$, we investigate the linear extremal problem of finding a function $F \in A^p$ with norm $\|F\|_{A^p} = 1$ for which

$$\text{Re } \phi(F) = \sup_{\|g\|_{A^p}=1} \text{Re } \phi(g) = \|\phi\|. \tag{1.2}$$

Such a function F is called an extremal function, and we say that F is an extremal function for a function $k \in A^q$ if F solves problem (1.2) for the functional ϕ with kernel k. This problem has been studied by numerous authors (see the introduction and references for some examples). Note that for $p = 2$, the extremal function is $F = k/\|k\|_{A^2}$.

A closely related problem is that of finding $f \in A^p$ such that $\phi(f) = 1$ and

$$\|f\|_{A^p} = \inf_{\phi(g)=1} \|g\|_{A^p}. \tag{1.3}$$

If F solves the problem (1.2), then $\frac{F}{\phi(F)}$ solves the problem (1.3), and if f solves (1.3), then $\frac{f}{\|f\|}$ solves (1.2). When discussing either of these problems, we always assume that ϕ is not the zero functional, in other words, that k is not identically 0.

It is well known that the problems (1.2) and (1.3) each have a unique solution when $1 < p < \infty$ (see e.g. [22], or [8], Theorem 1.4). Also, for every function $f \in A^p$ such that f is not identically 0, there is a unique $k \in A^q$ such that f solves problem (1.3) for k (see e.g. [8], Theorem 3.3). This implies that for each $F \in A^p$ with $\|F\|_{A^p} = 1$, there is some nonzero k such that F solves problem

(1.2) for k. Furthermore, any two such kernels k are positive multiples of each other.

The next result is an important characterization of extremal functions in A^p for $1 < p < \infty$ (see [23], p. 55).

Theorem A. *Let $1 < p < \infty$ and let $\phi \in (A^p)^*$. A function $F \in A^p$ with $\|F\|_{A^p} = 1$ and Re $\phi(F) > 0$ satisfies*

$$\textit{Re}\ \phi(F) = \sup_{\|g\|_{A^p}=1} \textit{Re}\ \phi(g) = \|\phi\|$$

if and only if

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = 0$$

for all $h \in A^p$ with $\phi(h) = 0$. If F satisfies the above conditions, then

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = \frac{\phi(h)}{\|\phi\|}$$

for all $h \in A^p$.

The following may also be found in [23], p. 55.

Theorem B. *Suppose that X is a closed subspace of $L^p(\mathbb{D})$, for $1 < p < \infty$. Let $F \in L^p$ and suppose that for all $h \in X$, we have $\|F\| \le \|F + h\|$. Then,*

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F} d\sigma = 0$$

for all $h \in X$.

Because point evaluation is a bounded linear functional on the Hilbert space A^2, the space A^2 has a reproducing kernel $K(z, \zeta)$, called the *Bergman kernel*, with the property that

$$f(z) = \int_{\mathbb{D}} K(z, \zeta) f(\zeta)\, d\sigma(\zeta) \tag{1.4}$$

for all $f \in A^2$ and for all $z \in \mathbb{D}$. One can show that

$$K(z, \zeta) = \frac{1}{(1 - \overline{\zeta} z)^2}.$$

Since the polynomials are dense in A^1, we have that (1.4) holds for all $f \in A^1$.

In fact, for any f in L^1 we many define the Bergman projection $\mathcal{P}$ by

$$(\mathcal{P}f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - \overline{\zeta} z)^2}\, d\sigma(\zeta).$$

The Bergman projection maps L^1 into the space of functions analytic in $\mathbb{D}$. A non-trivial fact is that $\mathcal{P}$ also maps L^p boundedly onto A^p for $1 < p < \infty$. If $p = 2$, then $\mathcal{P}$ is just the orthogonal projection of L^2 onto A^2.

The rest of this paper is organized as follows. In section 2, we prove various theorems relating the Bergman projection to extremal problems. In section 3, we calculate various Bergman projections. We use these results in section 4 to solve some extremal problems explicitly. Lastly, in section 5, we apply our results to the study of canonical divisors in A^p when p is an even integer.

2 Relation of the Bergman Projection to Extremal Problems

In this section we show how information about the Bergman projection can be used to solve certain extremal problems. We begin with a basic theorem that is obvious but quite useful.

Theorem 2.1. *Suppose that* $1 < p < \infty$ *and let* $f \in A^p$ *and* $g \in L^q$*, where* $1/p + 1/q = 1$*. Then*

$$\int_{\mathbb{D}} f\overline{g}\, d\sigma = \int_{\mathbb{D}} f\, \overline{\mathcal{P}(g)}\, d\sigma.$$

Proof. The case $p = 2$ follows from the fact that $\mathcal{P}$ is the orthogonal projection from L^2 onto A^2. The other cases follow from a routine approximation argument, using the fact that $\mathcal{P} : L^p \to A^p$ boundedly. □

The next theorem gives the first indication of how the Bergman projection is related to extremal problems.

Theorem 2.2. *Suppose that* $1 < p < \infty$*. Let* $F \in A^p$ *with* $\|F\|_{A^p} = 1$*. Then* F *is the extremal function for the functional with kernel*

$$k = \mathcal{P}(|F|^{p-1} \operatorname{sgn} F) \in A^q.$$

Furthermore, if F *is the extremal function for some functional* $\phi \in (A^p)^*$ *with kernel* $k \in A^q$*, then*

$$\mathcal{P}(|F|^{p-1} \operatorname{sgn} F) = \frac{k}{\|\phi\|}.$$

Proof. Consider the functional $\psi \in (A^p)^*$ that takes a function $f \in A^p$ to

$$\psi(f) = \int_{\mathbb{D}} f|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma.$$

This functional has norm at most $\|\, |F|^{p-1}\overline{\operatorname{sgn} F}\, \|_{L^q} = \|F\|_{L^p}^{p/q} = 1$. But also $\psi(F) = \|F\|_{A^p}^p = 1$, so ψ has norm exactly 1 and F is the extremal function for ψ.

But from Theorem 2.1, it follows that

$$\int_{\mathbb{D}} f\, \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)}\, d\sigma = \int_{\mathbb{D}} f|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma$$

for any $f \in A^p$, which means that $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is the kernel in A^q representing ψ. This proves the first part of the theorem.

If F is the extremal function for ϕ, then ψ is a positive scalar multiple of ϕ (see Section 1.) Since $\|\psi\| = 1$ and ψ is a positive scalar multiple of ϕ, it must be that $\psi = \phi/\|\phi\|$. But this implies that $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F) = k/\|\phi\|$. □

The next result, Theorem 2.4, describes the relation of the Bergman projection to a sort of generalized minimal interpolation problem. The problem is to find the function of smallest norm such that prescribed linear functionals acting on the function take prescribed values. We will first need the following lemma.

Lemma 2.3. *Let V be a vector space over $\mathbb{C}$, and let $\phi, \phi_1, \ldots, \phi_N$ be linear functionals on V such that, for $v \in V$, if $\phi_1(v) = \cdots = \phi_N(v) = 0$, then $\phi(v) = 0$. Then $\phi = \sum_{j=1}^N c_j \phi_j$ for some constants c_j.*

The statement and proof of this lemma may be found in [3] in Appendix A.2 as Proposition 1.4.

Theorem 2.4. *Let $1 < p < \infty$ and let $\phi_1, \phi_2, \ldots, \phi_N \in (A^p)^*$ be linearly independent. Then a function $F \in A^p$ satisfies*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}$$

if and only if $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \ldots, \phi_N$.

Note that this theorem gives a necessary and sufficient condition for a function F to solve the minimal interpolation problem of finding a function $f \in A^p$ of smallest norm such that $\phi_j(f) = c_j$ for $1 \le j \le N$, where $\phi_j \in (A^p)^*$ are arbitrary linearly independent functionals and the c_j are given constants. Namely, F solves the problem if and only if $\phi_j(F) = c_j$ for $1 \le j \le N$ and $\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \ldots, \phi_N$. Note that for the case $1 < p < \infty$, the problem under discussion will always have a unique solution (see e.g. [8], Proposition 1.3).

Proof. Let $k_1, \ldots, k_N$ be the kernels of $\phi_1, \ldots, \phi_N$, respectively. Suppose that

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}$$

and let h be any non-zero A^p function such that $\phi_1(h) = \cdots = \phi_N(h) = 0$. Since there are only a finite number of the ϕ_j, it is clear that such a function exists. Then $F + h$ is also in contention to solve the extremal problem, so $\|F\| \le \|F + h\|$. Now Theorem B shows that

$$\int_{\mathbb{D}} |F|^{p-1} \overline{\operatorname{sgn} F}\, h \, d\sigma = 0,$$

and so by Theorem 2.1

$$\int_{\mathbb{D}} \overline{\mathcal{P}(|F|^{p-1} \operatorname{sgn} F)}\, h \, d\sigma = 0.$$

Define

$$\psi(f) = \int_{\mathbb{D}} \overline{\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)}\, f\, d\sigma, \qquad f \in A^p.$$

Lemma 2.3 now shows that

$$\psi = \sum_{j=1}^{N} c_j \phi_j,$$

for some constants c_j, so $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of $k_1, \ldots, k_n$. This proves the "only if" part of the theorem.

Conversely, suppose $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of $k_1, \ldots, k_n$. Now

$$\|F\|_{A^p}^p = \int_{\mathbb{D}} F|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma = \int_{\mathbb{D}} F\, \overline{\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)}\, d\sigma, \tag{2.1}$$

by Theorem 2.1. Now let $h \in A^p$ be such that $\phi_j(h) = 0$ for $1 \le j \le N$. Since $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the k_j, equation (2.1) gives

$$\begin{aligned}
\|F\|_{A^p}^p &= \int_{\mathbb{D}} (F+h)\overline{\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)}\, d\sigma \\
&= \int_{\mathbb{D}} (F+h)|F|^{p-1}\overline{\operatorname{sgn} F}\, d\sigma \\
&\le \|F+h\|_{A^p} \||F|^{p-1}\overline{\operatorname{sgn} F}\|_{A^q} \\
&= \|F+h\|_{A^p} \|F\|_{A^p}^{p-1}.
\end{aligned}$$

Therefore,

$$\|F\|_{A^p} \le \|F+h\|_{A^p}.$$

Since h was an arbitrary A^p function with the property that $\phi_j(h) = 0$ for $1 \le j \le N$, this shows that F solves the extremal problem in question. $\square$

When we apply this theorem, we will usually have each ϕ_j be a derivative-evaluation functional. By derivative-evaluation functional, we mean a functional defined by $f \mapsto f^{(n)}(z_0)$ for some integer $n \ge 0$ and some $z_0 \in \mathbb{D}$. Note that the theorem implies that, if $\phi_1, \phi_2, \ldots, \phi_N \in (A^p)^*$ are linearly independent, then the following two statements are equivalent:
1. F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \ldots, \phi_N(f) = \phi_N(F)\}$$

but does not satisfy

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_{j_1}(f) = \phi_{j_1}(F), \ldots, \phi_{j_M}(f) = \phi_{j_M}(F)\}$$

for any proper subsequence $\{j_k\}_{k=1}^M$ of $1, 2, \ldots, N$.
2. $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a linear combination of the kernels of $\phi_1, \ldots, \phi_N$, and none of the coefficients in the linear combination is 0.

The next theorem is a special case of Theorem 2.4, with the functionals taken to be $\phi_j(h) = h^{(j)}(0)$, with kernels $k_j(z) = (j+1)!z^j$.

Theorem 2.5. *The function $\mathcal{P}(|F|^{p-1}\operatorname{sgn} F)$ is a polynomial of degree at most N if and only if*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f(0) = F(0), \dots, f^{(N)}(0) = F^{(N)}(0)\}.$$

It is a polynomial of degree exactly N if and only if N is the smallest integer such that the above conditions holds.

The next theorem relates the generalized minimal interpolation problems we have been discussing with linear extremal problems.

Theorem 2.6. *Let $\phi_1, \dots, \phi_N$ be linearly independent elements of $(A^p)^*$ with kernels $k_1, \dots, k_N$ respectively, and let $F \in A^p$ with $\|F\|_{A^p} = 1$. Then the functional for which F is the extremal function has as its kernel a linear combination of the k_j if and only if*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \dots, \phi_N(f) = \phi_N(F)\}.$$

This follows from Theorems 2.2 and 2.4. Recall that although there is no unique functional for which F is the extremal function, such a functional is unique up to a positive scalar multiple, which does not affect whether its kernel is a linear combination of the k_j.

One direction of this theorem, the fact that if F is the extremal function for some kernel which is a linear combination of the k_j, then F solves the stated minimal interpolation problem, is easy to prove directly. The proof is as follows. Let F be the extremal function for the functional ϕ, which we assume to have kernel $k = \sum_{j=1}^{N} a_j k_j$. Then

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi(f) = \phi(F)\}.$$

But if some function G in A^p satisfies $\phi_j(F) = \phi_j(G)$ for all j with $1 \le j \le N$, then $\phi(G) = \phi(F)$, which implies that $\|F\|_{A^p} \le \|G\|_{A^p}$. This implies that F satisfies

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : \phi_1(f) = \phi_1(F), \dots, \phi_N(f) = \phi_N(F)\}.$$

3 Calculating Bergman Projections

Now that we have explored the relation between the Bergman projection and solutions to extremal problems, we will calculate the Bergman projection in various cases.

Proposition 3.1. *Let m and n be nonnegative integers. Then*

$$\mathcal{P}(z^m \overline{z}^n) = \begin{cases} \frac{m-n+1}{m+1} z^{m-n}, & \text{if } m \ge n, \\ 0, & \text{if } m < n. \end{cases}$$

This is Lemma 6 in Chapter 2 of [7].

The next theorem is very helpful in calculating the Bergman projection of the kernel of a derivative-evaluation functional times the conjugate of an A^p function.

Theorem 3.2. *Let* $1 < q_1, q_2 \leq \infty$*. Let* p_1 *and* p_2 *be the conjugate exponents of* q_1 *and* q_2*. Let*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$$

and suppose that $1 < q < \infty$*. Let* p *be the conjugate exponent of* q*. Suppose that* $k \in A^{q_1}$ *and that* $g \in A^{q_2}$*. Let the functional* ψ *be defined by* $\psi(f) = \int_{\mathbb{D}} f\overline{k}\, d\sigma$ *for all* $f \in A^{p_1}$*. Then* $\mathcal{P}(k\overline{g})$ *is the kernel of the functional* $\phi \in (A^p)^*$ *defined by*

$$\phi(f) = \psi(fg), \qquad f \in A^p.$$

Proof. First note that $1/p + 1/q_1 + 1/q_2 = 1$, so if $f \in A^p$, then $fg \in A^{p_1}$ and the definition of ϕ makes sense. Now observe that

$$\phi(f) = \psi(fg) = \int_{\mathbb{D}} fg\overline{k}\, d\sigma.$$

By Theorem 2.1, this equals

$$\int_{\mathbb{D}} f\overline{\mathcal{P}\left(\overline{g}k\right)}\, d\sigma. \qquad \square$$

We will study the kernels of various derivative-evaluation functionals. Evaluation at the origin is somewhat different and simpler than evaluation elsewhere, so we deal with it first.

Theorem 3.3. *The kernel for the functional* $f \mapsto f^{(n)}(0)$ *is* $(n+1)!z^n$*. If* $g \in A^p$ *then*

$$\mathcal{P}(z^n\overline{g(z)}) = \sum_{j=0}^{n} \frac{\overline{g^{n-j}(0)}}{(n-j)!} \frac{j+1}{n+1} z^n.$$

Proof. The first statement can be verified by evaluating

$$\int_{\mathbb{D}} f(z)\overline{z}^n d\sigma(z)$$

when f is written as a power series. The second part follows from Proposition 3.1. To see this, note that by the first part of the theorem, $\mathcal{P}(z^n\overline{g(z)})$ is the kernel for the functional taking $f \in A^p$ to

$$\frac{1}{(n+1)!}(fg)^{(n)}(0) = \frac{1}{(n+1)!} \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(0) g^{(n-j)}(0),$$

which has kernel

$$\sum_{j=0}^{n} \frac{\overline{g^{n-j}(0)}}{(n-j)!} \frac{j+1}{n+1} z^j. \qquad \square$$

We will now deal with the function $1/(1-\overline{a}z)^n$, for $n \geq 2$.

Proposition 3.4. *The kernel for the functional* $f \mapsto f^{(n)}(a)$ *is*

$$\frac{(n+1)!z^n}{(1-\overline{a}z)^{n+2}}, \qquad n = 0, 1, 2, \ldots$$

Proof. We know that

$$f(a) = \int_{\mathbb{D}} \frac{1}{(1-a\overline{z})^2} f(z)\, d\sigma(z).$$

Differentiation n times with respect to a gives the result. □

Proposition 3.5. *For each* $a \in \mathbb{D}$ *with* $a \neq 0$*, there are numbers* $c_0, \ldots, c_n$ *with* $c_n \neq 0$ *such that the function*

$$\frac{1}{(1-\overline{a}z)^{n+2}}$$

is the kernel for the functional $f \mapsto c_0 f(a) + c_1 f'(a) + \ldots + c_n f^{(n)}(a)$.

Proof. We will proceed by induction. The claim is true for $n = 0$ by the reproducing property of the Bergman kernel function. For general n, we may write the partial fraction expansion

$$\frac{z^n}{(1-\overline{a}z)^{n+2}} = \sum_{j=0}^{n+2} \frac{b_j}{(1-\overline{a}z)^j},$$

for some complex numbers b_j. Thus,

$$z^n = \sum_{j=0}^{n+2} b_j (1-\overline{a}z)^{n+2-j}.$$

Differentiating both sides $n+1$ times with respect to z gives

$$0 = b_1(-\overline{a})^{n+1}(n+1)! + b_0(-\overline{a})^{n+1}(n+2)!(1-\overline{a}z).$$

Since this holds for all z, it follows that $b_0 = b_1 = 0$. Since $z^n/(1-\overline{a}z)^{n+2}$ has a pole of order $n+2$ at $1/\overline{a}$, we see that $b_{n+2} \neq 0$. Therefore,

$$\frac{1}{(1-\overline{a}z)^{n+2}} = \frac{1}{b_{n+2}} \left(\frac{z^n}{(1-\overline{a}z)^{n+2}} - \sum_{j=2}^{n+1} \frac{b_j}{(1-\overline{a}z)^j} \right).$$

Note that the first term of the right side of the above equation is the kernel for the functional $f \mapsto (1/(n+1)!)f^{(n)}(a)$. Also, each term in the sum $\sum_{j=2}^{n+1} \frac{b_j}{(1-\overline{a}z)^j}$ is the kernel for a linear functional taking each function f to some linear combination of $f(a), f'(a), \ldots,$ and $f^{(n-1)}(a)$, by the induction hypothesis. This proves the proposition. □

Proposition 3.6. *Let $g \in A^p$ for $1 < p < \infty$, and let $a \in \mathbb{D}$ with $a \neq 0$. Suppose g has a zero of order n at a. Let $N \geq 0$ be an integer. Then*

$$\mathcal{P}\left(\frac{1}{(1-\overline{a}z)^{N+2}}\overline{g(z)}\right) = \sum_{k=0}^{N-n} C_k \frac{1}{(1-\overline{a}z)^{k+2}}$$

for some complex constants C_k depending on $g^{(m)}(a)$ for $0 \leq m \leq N$.

Proof. The projection

$$\mathcal{P}\left(\frac{1}{(1-\overline{a}z)^{N+2}}\,\overline{g(z)}\right)$$

is the kernel associated with the functional

$$f \mapsto \sum_{j=0}^{N} b_j (fg)^{(j)}(a)$$

for some constants b_j, with $b_N \neq 0$, by the previous proposition and Theorem 3.2. But

$$\sum_{j=0}^{N} b_j (fg)^{(j)}(a) = \sum_{j=0}^{N}\sum_{k=0}^{j} b_j \binom{j}{k} f^{(k)}(a) g^{(j-k)}(a).$$

Since $g^{(j)}(a) = 0$ for $0 \leq j < n$, all terms in the sum with $j - k < n$ are 0. But this means that the only non-zero terms in the sum occur when $k \leq j - n$, so that $k \leq N - n$. Now, set

$$B_k = \sum_{j=k+n}^{N} b_j \binom{j}{k} g^{(j-k)}(a),$$

so

$$\sum_{j=0}^{N} b_j (fg)^{(j)}(a) = \sum_{k=0}^{N-n} B_k f^{(k)}(a).$$

But the kernel associated to $\sum_{k=0}^{N-n} B_k f^{(k)}(a)$ is

$$\sum_{k=0}^{N-n} B_k \frac{(k+1)!z^k}{(1-\overline{a}z)^{k+2}}.$$

As in the proof of Theorem 3.5, we may show that

$$\frac{z^k}{(1-\overline{a}z)^{k+2}} = \frac{c_{k2}}{(1-\overline{a}z)^2} + \frac{c_{k3}}{(1-\overline{a}z)^3} + \cdots + \frac{c_{k,k+2}}{(1-\overline{a}z)^{k+2}}$$

for some constants $c_{k2}, \ldots, c_{k,k+2}$. Thus we may write

$$\sum_{k=0}^{N-n} B_k \frac{(k+1)!z^k}{(1-\overline{a}z)^{k+2}} = \sum_{k=0}^{N-n} C_k \frac{1}{(1-\overline{a}z)^{k+2}}$$

for some constants C_k, depending on $g^{(m)}(a)$ for $0 \leq m \leq N$. □

We will now deal with the function $1/(1-\overline{a}z)$. Since the functional with kernel $1/(1-\overline{a}z)^{n+2}$ involves differentiation of order n, it seems reasonable that the functional with kernel $1/(1-\overline{a}z)$ involves integration. This is indeed the case.

Proposition 3.7. *The function*

$$1/(1-\overline{a}z)$$

is the kernel for the functional defined on A^p *for* $1 < p < \infty$ *by*

$$f \mapsto \frac{1}{a}\int_0^a f(z)\,dz.$$

Proof. Since

$$\frac{1}{1-\overline{a}z} = \sum_{n=0}^{\infty} (\overline{a}z)^n,$$

it follows that

$$\int_{\mathbb{D}} \frac{z^m}{1-a\overline{z}}\,d\sigma = \sum_{n=0}^{\infty}\int_{\mathbb{D}} (a\overline{z})^n z^m\,d\sigma = a^m \int_{\mathbb{D}} |z|^{2m}\,d\sigma = \frac{a^m}{m+1}. \tag{3.1}$$

The change in the order of integration and summation is justified by the fact that the sum converges uniformly in $\overline{\mathbb{D}}$. Now let $f \in A^p$ and write $f(z) = \sum_{m=0}^{\infty} b_m z^m$. Define

$$F(z) = \frac{1}{z}\int_0^z f(\zeta)\,d\zeta = \sum_{m=0}^{\infty} \frac{b_m}{m+1} z^m.$$

Therefore, by equation (3.1),

$$\int_{\mathbb{D}} \frac{1}{1-a\overline{z}} f(z)\,d\sigma = \int_{\mathbb{D}} \frac{1}{1-a\overline{z}} \left(\sum_{m=0}^{\infty} b_m z^m\right) d\sigma = \sum_{m=0}^{\infty} b_m \frac{a^m}{m+1} = F(a).$$

The interchange of the order of integration and summation is justified by the fact the partial sums of the Taylor series for f approach f in A^p. □

The following theorem is quite useful for determining what form certain Bergman projections have.

Theorem 3.8. *For* $1 \le n \le N$*, let* d_n *be a nonnegative integer and let* $z_n \in \mathbb{D}$*. Let* k *be analytic and a linear combination of the kernels of the functionals given by* $f \mapsto f^{(d_n)}(z_n)$*. Let* $g \in A^p$ *for* $p > 1$*. Then* $\mathcal{P}(k\overline{g})$ *is in the linear span of the set of all the kernels of functionals defined by* $f \mapsto f^{(m)}(z_n)$*, where* m *is an integer with* $0 \le m \le d_n$ *and* n *is an integer with* $1 \le n \le N$*.*

Proof. Let $k = \sum_{n=1}^{N} a_n k_n$, where k_n is the kernel for the functional $f \mapsto f^{(d_n)}(z_n)$. Then by Theorem 3.2, $\mathcal{P}(k_n\overline{g})$ is the kernel of the functional

$$f \mapsto (fg)^{(d_n)}(z_n) = \sum_{j=0}^{d_n} \binom{d_n}{j} f^{(j)}(z_n) g^{(d_n-j)}(z_n).$$

But this functional is a linear combination of functionals of the form

$$f \mapsto f^{(m)}(z_n),$$

where $0 \le m \le d_n$. □

Due to their relation with extremal problems, we are often concerned with projections of the form $\mathcal{P}(F^{p/2}\overline{F}^{(p/2)-1})$, where F is an analytic function. This is well defined because

$$F^{p/2}\overline{F}^{(p/2)-1} = |F|^p/\overline{F} = |F|^{p-1} \operatorname{sgn} F.$$

The following theorems deal with this situation.

Theorem 3.9. *Let $1 < p < \infty$, and let $F \in A^p$. Furthermore, suppose if $p < 2$ that $F^{(p/2)-1} \in A^{p_1}$ for some $p_1 > 1$. Also, suppose that $F^{p/2}$ is analytic and is a linear combination of the kernels k_n corresponding to the functionals $f \mapsto f^{(d_n)}(z_n)$ for some integers d_n and some points $z_n \in \mathbb{D}$, where $1 \le n \le N$, and where N is an integer. Then F satisfies*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f^{(m)}(z_n) = F^{(m)}(z_n) \text{ for all } n \text{ and } m \text{ such that } 1 \le n \le N \text{ and } 1 \le m \le d_n\}.$$

Proof. This follows from Theorems 3.8 and 2.4. □

The following theorem is a consequence of Theorem 3.8. It can also be proved by using Taylor series and Proposition 3.1.

Theorem 3.10. *Let f be a polynomial of degree at most N and let $g \in A^p$ for some $p > 1$. Then $\mathcal{P}(f\overline{g})$ is a polynomial of degree at most N.*

Using this theorem and Theorem 2.5, we immediately get the following result.

Theorem 3.11. *Suppose that $F \in A^p$ and $F^{p/2}$ is a polynomial of degree N. Furthermore, if $p < 2$ suppose that $F^{(p/2)-1} \in A^{p_1}$ for some $p_1 > 1$. Then*

$$\|F\|_{A^p} = \inf\{\|f\|_{A^p} : f(0) = F(0), \ldots, f^{(N)}(0) = F^{(N)}(0)\}.$$

Note that $F^{p/2}$ can be a polynomial only if F is nonzero in $\mathbb{D}$ or $p/2$ is rational and all the zeros of F in $\mathbb{D}$ are of order a multiple of s, where r/s is the reduced form of $p/2$. If p is an even integer, this poses no restriction. Because of this, the case where p is an even integer is often easier to work with.

4 Solution of Specific Extremal Problems

We will now discuss how to solve some specific minimal interpolation problems. Since we are dealing with the powers $p/2$ and $2/p$, neither of which need be an integer, we will have to take care in our calculations. We will introduce a lemma to facilitate this. The lemma basically says that if f and g are analytic functions nonzero at the origin, and if $f^{(n)}(0) = (g^p)^{(n)}(0)$ for all n such that $0 \le n \le N$, then $(f^{1/p})^{(n)}(0) = g^{(n)}(0)$ for all n such that $0 \le n \le N$.

To state the lemma we first need to introduce some notation. Suppose that the constants $c_0, c_1, \ldots, c_N$ are given and that $c_0 \neq 0$, and let $h(z) = c_0 + c_1 z + \cdots + c_N z^N$. Suppose that $a = c_0^p$ for some branch of the function z^p. Let U be a neighborhood of the origin such that $h(U)$ is contained in some half plane whose boundary contains the origin, and such that $0 \notin h(U)$. Then we can define z^p so that it is analytic in $h(U)$ and so that $c_0^p = a$. We let $\beta_j^p(a; c_0, c_1, \ldots, c_j)$ denote the j^{th} derivative of $h(z)^p$ at 0.

Note that because of the chain rule for differentiation, β_j^p only depends on j, the constants $c_0, \ldots, c_j$, and the numbers p and a. For the same reason, the value of β_j^p is the same if we replace the function h in the definition of β_j^p by any function $\widetilde{h}$ analytic near the origin such that $\widetilde{h}^{(j)}(0) = c_j$ for $1 \le j \le N$.

Lemma 4.1. *Let $c_0, c_1, \ldots, c_N$ be given complex numbers, and let p be a real number. Suppose that $c_0 \neq 0$, and let $a_0 = c_0^p$, for some branch of z^p. Then*

$$c_j = \beta_j^{1/p}\left(c_0; \beta_0^p(a_0; c_0), \beta_1^p(a_0; c_0, c_1), \ldots, \beta_j^p(a_0; c_0, \ldots, c_j)\right).$$

Proof. Let $a_j = \beta_j^p(a_0; c_0, \ldots, c_j)$ and $b_j = \beta_j^{1/p}(c_0; a_0, \ldots, a_j)$. Then $b_0 = c_0$.

Now let $f(z) = \sum_{j=0}^N \frac{c_j}{j!} z^j$. Then $f^{(j)}(0) = c_j$ for $0 \le j \le N$. Let U be a neighborhood of 0 such that there exist $r_0 > 0$ and $\theta_0 \in \mathbb{R}$ such that

$$f(U) \subset \left\{ re^{i\theta} : r_0 < r \text{ and } \theta_0 - \frac{\pi}{2p} < \theta < \theta_0 + \frac{\pi}{2p} \right\}.$$

Then z^p can be defined as an analytic function in $f(U)$. Furthermore, the set $V = (f(U))^p$ does not contain 0 but is contained in some half plane, so $z^{1/p}$ can be defined as an analytic function in V so that it is the inverse of the function z^p defined in $f(U)$.

Now define $g(z) = (f(z))^p$ for $z \in U$. Then $g^{(j)}(0) = a_j$ and $g^{1/p}(0) = c_0$, so $(g^{1/p})^{(j)}(0) = b_j$ for $0 \le j \le N$. But $g(z)^{1/p} = f(z)$ for $z \in U$, so $b_j = c_j$ for $0 \le j \le N$. □

We will now use the lemma to solve a specific extremal problem in certain cases.

Theorem 4.2. *Let $c_0, \ldots, c_N$ be given complex numbers, and assume that $c_0 \neq 0$. Suppose that $F \in A^p$, and $F^{(j)}(0) = c_j$ for $0 \le j \le N$, and*

$$\|F\|_{A^p} = \inf\{\|g\|_{A^p} : g(0) = c_0, \ldots, g^{(N)}(0) = c_N\}.$$

Let $a_0 = c_0^{p/2}$ *for some branch of* z^p*. Define*

$$f(z) = \sum_{j=0}^{N} \frac{\beta_j^{p/2}(a_0; c_0, \dots, c_j)}{j!} z^j,$$

where the $\beta_j^{p/2}$ *are defined as in the beginning of this section. Suppose that* $f^{1-(2/p)} \in A^{p_1}$ *for some* $p_1 > 1$*. Also, suppose that* f *has no zeros in* $\mathbb{D}$*. Thus we may define* $f^{2/p}$ *so that it is analytic in* $\mathbb{D}$ *and so that* $f^{2/p}(0) = c_0$*. Then*

$$F = f^{2/p}.$$

The same result also holds if p *is rational,* $2/p = r/s$ *in lowest form, and every zero of* f *has order a multiple of* s*.*

Proof. Note that $f^{2/p}$ is analytic in $\mathbb{D}$ since we have assumed f has no zeros in $\mathbb{D}$, or that p is rational and $2/p = r/s$ in lowest form and f has only zeros whose orders are multiples of s. Also, $f(0) = a_0$, so we may define $f^{2/p}$ so that $f^{2/p}(0) = c_0$. The j^{th} derivative of $f^{2/p}$ at 0 is

$$\beta_j^{2/p}\left(c_0; \beta_0^{p/2}(a_0; c_0), \dots, \beta_j^{p/2}(a_0; c_0, \dots, c_j)\right)$$

for $0 \le j \le N$, which equals c_j by the lemma. Thus $f^{2/p}$ is in contention to solve the extremal problem.

But

$$\mathcal{P}\left(\frac{|f^{2/p}|^p}{\overline{f}^{2/p}}\right) = \mathcal{P}(f\overline{f}^{1-(2/p)})$$

is a polynomial of degree at most N by Theorem 3.10, so by Theorem 3.11 we find $F = f^{2/p}$. □

To apply this theorem, we need to show that f has no zeros in the unit disc, or has only zeros of suitable orders if p is rational. Then, as long as $f^{1-(2/p)} \in A^{p_1}$ for some $p_1 > 1$, we have that $f^{2/p}$ is the extremal function. Note that we do not need to know anything about the zeros of the extremal function itself to apply the theorem, but only about the zeros of f.

Also note that if f has no zeros in the unit disc, this theorem implies that the extremal function $F = f^{2/p}$ also has no zeros in the unit disc. It can also be shown that if F has no zeros, then F must equal $f^{2/p}$. This follows from [13], Theorem B. It also follows from the work on extremal problems in Bergman space posed over non-vanishing functions found in [1], [24], [26], and [25]. The case where the extremal function has zeros is more challenging and not as well understood. See Example 4.6 for a problem in which the extremal function has one zero.

Example 4.3. The solution to the minimal interpolation problem in A^p with $f(0) = 1$ and $f'(0) = c_1$ is

$$F(z) = \left(1 + \frac{p}{2} c_1 z\right)^{2/p},$$

provided that $|c_1| \le \frac{2}{p}$ (or $p = 2$). This is because $\beta_0^{p/2}(1;1,c_1) = 1$ and $\beta_1^{p/2}(1;1,c_1) = (p/2)c_1$. For example, if $p = 4$ and $c_1 = \frac{1}{2}$, then

$$F(z) = (1+z)^{1/2}.$$

The above problem is also solved in [19] in more general form. The solution to the extremal problem in the next example is more difficult. We do not know if it has been stated explicitly before, although in the case in which the extremal function has no zeros it does follow from [13], Theorem B, or from the results in [24], if it is assumed *a priori* that the extremal function has no zeros.

Example 4.4. The solution to the minimal interpolation problem in A^p for $1 < p < \infty$ with $F(0) = 1$, and $F'(0) = c_1$, and $F''(0) = c_2$ is

$$F(z) = \left\{1 + (p/2)c_1 z + [(p(p-2)/4)c_1^2 + (p/2)c_2]z^2\right\}^{2/p},$$

provided that the quadratic polynomial under the $2/p$ exponent in the equation for F has no zeros in $\mathbb{D}$. The solution is the same if $p = 4$ or $4/3$ and the quadratic polynomial has a repeated root in the unit disc. (The solution is also the same in the case $p = 2$, no matter where the polynomial has roots).

Linear extremal problems tend to be more difficult to solve than minimal interpolation problems involving derivative-evaluation functionals, because values of a function f and its derivatives are generally easier to calculate than $\mathcal{P}(|f|^{p-1}\operatorname{sgn} f)$. Nevertheless it is possible to solve some linear extremal problems explicitly by the methods in this paper. Here is one example.

Theorem 4.5. *Let $N \ge 1$ be an integer, let $1 < p < \infty$, and let $b \in \mathbb{C}$ satisfy*

$$|b| \ge 1 + \frac{1}{N+1}\left(1 - \frac{2}{p}\right),$$

and define

$$a = \frac{|b| + \sqrt{|b|^2 - \frac{4}{N+1}\left(1 - \frac{2}{p}\right)}}{2}\operatorname{sgn} b.$$

Then the solution to the extremal problem in A^p with kernel $z^N + b$ is

$$F(z) = \operatorname{sgn}(a^{1-(2/p)})\frac{(z^N + a)^{2/p}}{\left(|a|^2 + 1/(N+1)\right)^{1/p}}.$$

In the above expression for $F(z)$, the branch of $(z^N + a)^{2/p}$ may be chosen arbitrarily, but the value of $\operatorname{sgn}(a^{1-(2/p)})$ must be chosen consistently with this choice. Note that the functional associated with the kernel $z^N + b$ is

$$\phi(f) = bf(0) + (1/(N+1)!)f^{(N)}(0).$$

Also, observe that the hypothesis of the theorem holds for all N and p if $|b| \ge 3/2$.

Proof. The condition

$$|b| \geq 1 + \frac{1}{N+1}\left(1 - \frac{2}{p}\right)$$

implies that $|a| \geq 1$, so that $z^N + a \neq 0$ in $\mathbb{D}$ and F is an analytic function. Note that

$$\|(z^N + a)^{2/p}\|_{A^p}^p = \int_{\mathbb{D}} |z^N + a|^2 \, d\sigma = \int_{\mathbb{D}} (z^N + a)\overline{(z^N + a)} \, d\sigma = |a|^2 + \frac{1}{N+1}.$$

Thus, $\|F\|_{A^p} = 1$.

Now

$$((z^N + a)^{2/p})^{p/2-1} = a^{1-2/p} + \left(1 - \frac{2}{p}\right) a^{-2/p} z^N + O(z^{2N}),$$

where we choose branches so that $((z^N + a)^{2/p})^{p/2} = z^N + a$. We calculate that

$$\begin{aligned} &\mathcal{P}\left(|z^N + a|^{p-1} \operatorname{sgn}(z^N + a)\right) \\ &= \mathcal{P}\left((z^N + a)\overline{(z^N + a)}^{1-2/p}\right) \\ &= \mathcal{P}\left((z^N + a)\left(\overline{a^{1-2/p}} + \left(1 - \frac{2}{p}\right)\overline{a^{-2/p}}\,\overline{z^N} + O(\overline{z^{2N}})\right)\right). \end{aligned}$$

But by Proposition 3.1, this equals

$$\begin{aligned} &\mathcal{P}\left[(z^N + a)\left(\overline{a^{1-2/p}} + \left(1 - \frac{2}{p}\right)\overline{a^{-2/p}}\,\overline{z^N}\right)\right] \\ &= a\overline{a^{1-2/p}} + \frac{1}{N+1}\left(1 - \frac{2}{p}\right)\overline{a^{-2/p}} + \overline{a^{1-(2/p)}} z^N \\ &= \overline{a^{1-(2/p)}}\left(z^N + a + \frac{1}{N+1}\left(1 - \frac{2}{p}\right)\overline{a}^{\,-1}\right) \\ &= \overline{a^{1-(2/p)}}(z^N + b). \end{aligned}$$

Thus,

$$\begin{aligned} &\mathcal{P}\left\{\left|\operatorname{sgn}(a^{1-(2/p)})(z^N + a)\right|^{p-1} \operatorname{sgn}\left(\operatorname{sgn}(a^{1-(2/p)})(z^N + a)\right)\right\} \\ &= \overline{a^{1-(2/p)}} \operatorname{sgn}(a^{1-(2/p)})(z^N + b) \\ &= |a^{1-(2/p)}|\,(z^N + b). \end{aligned}$$

Therefore,

$$\mathcal{P}(F^{p/2}\overline{F^{(p/2)-1}}) = |a^{1-(2/p)}| \frac{z^N + b}{(|a|^2 + 1/(N+1))^{(p-1)/p}}.$$

Since $\|F\|_{A^p} = 1$, Theorem 2.2 shows that F is the extremal function for the kernel on the right of the above equation. But that kernel is a positive scalar multiple of k, so F is also the extremal function for k. □

Example 4.6. Let $a \in \mathbb{D} \setminus \{0\}$ and let $b, c \in \mathbb{C}$. Consider the function

$$f(z) = \frac{1}{a} \frac{a-z}{1-\overline{a}z} \left(1 + bz + cz^2\right)^{1/2},$$

where we assume that $1 + bz + cz^2$ has no zeros in $\mathbb{D}$, or a double zero in $\mathbb{D}$ (so that f is analytic). We choose the branch of this function so that $f(0) = 1$. Then a calculation shows that

$$\begin{aligned} f'(0) &= \frac{1}{a}\left(\frac{ab}{2} + |a|^2 - 1\right), \text{ and} \\ f''(0) &= \frac{1}{a}\left(|a|^2 b + 2a\overline{a}^2 + ac - 2\overline{a} - b - \frac{ab^2}{4}\right). \end{aligned}$$

Another calculation shows that the residue of f^2 about $\overline{a^{-1}}$ is equal to

$$|a|^{-4}\overline{a}^{-3}\left[(|a|^2-1)^2(2c+\overline{a}b) - 2(|a|^2-1)(\overline{a}^2 + c + \overline{a}b)\right].$$

Now, suppose that v_1 and v_2 are complex numbers, and that we want to solve the minimal interpolation problem of finding $F \in A^4$ such that $F(0) = 1, F'(0) = v_1, F''(0) = v_2$, and with $\|F\|_4$ as small as possible. If we can find numbers a, b, and c so that $(1 + bz + cz^2)$ has no zeros in $\mathbb{D}$, or a repeated zero, and so that

$$\begin{aligned} v_1 &= \frac{1}{a}\left(\frac{ab}{2} + |a|^2 - 1\right) \\ v_2 &= \frac{1}{a}\left(|a|^2 b + 2a\overline{a}^2 + ac - 2\overline{a} - b - \frac{ab^2}{4}\right) \\ 0 &= (|a|^2-1)^2(2c+\overline{a}b) - 2(|a|^2-1)(\overline{a}^2 + c + \overline{a}b) \end{aligned}$$

then

$$F(z) = f(z) = \frac{1}{a} \frac{a-z}{1-\overline{a}z} \left(1 + bz + cz^2\right)^{1/2}.$$

To see this, note that in this case

$$f(z)^2 = a_1 z^2 + a_2 z + a_3 + \frac{a_4}{(1-\overline{a}z)^2}$$

for some constants $a_1, \ldots, a_4$. Then

$$\mathcal{P}\left[\left(a_1 z^2 + a_2 z + a_3\right)\overline{f(z)}\right]$$

will be a polynomial of degree at most two by Theorem 3.10. Also, since $f(a) = 0$, we have

$$\mathcal{P}\left[\frac{a_4}{(1-\overline{a}z)^2}\overline{f(z)}\right] = 0$$

by Proposition 3.6. Thus, $\mathcal{P}(f^2\overline{f})$ is a polynomial, and so f solves the extremal problem in question, by Theorem 2.5.

5 Canonical Divisors

We will now discuss how our previous results apply to canonical divisors. These divisors are the Bergman space analogues of Blaschke products. They were first introduced in the A^2 case in [11], and were further studied for general p in [6] and [4]. The formula for a canonical divisor with one zero is well known, see for example [7]. In [10], a formula was obtained for canonical divisors with two zeros, as well as with more zeros under certain symmetry conditions on the zeros. In [16], a method is given for finding the canonical divisor in A^2 for an arbitrary finite zero set. In [17], a fairly explicit formula for canonical divisors is obtained for general p. In this section, we discuss how the methods of this paper apply to the problem of finding canonical divisors in the case where p is an even integer. The results we obtain are similar to those in [17].

By the zero-set of an A^p function not identically 0, we mean its collection of zeros, repeated according to multiplicity. Such a set will be countable, since the zeros of analytic functions are discrete. Given an A^p zero set, we can consider the space N^p of all functions that vanish on that set. More precisely, $f \in A^p$ is in N^p if it vanishes at every point in the given zero set, to at least the prescribed multiplicity.

If the zero set does not include 0, we pose the extremal problem of finding $G \in N^p$ such that $\|G\|_{A^p} = 1$, and such that $G(0)$ is positive and as large as possible. If the zero set has a zero of order n at 0, we instead maximize $G^{(n)}(0)$. For $0 < p < \infty$, this problem has a unique solution, which is called the canonical divisor. For $1 < p < \infty$, this follows from the fact that an equivalent problem is to find an $F \in N^p$ with $F(0) = 1$ and $\|F\|_{A^p}$ as small as possible. It is well known that the latter problem has a unique solution (see e.g. [8], Proposition 1.3).

In this section, we discuss the problem of determining the canonical divisor when p is an even integer, and the zero set is finite. We show how the methods of this paper can be used to characterize the canonical divisor. Our methods show that if G is the canonical divisor, then $G^{p/2}$ is a rational function with residue 0 at each of its poles, which is the content of the following theorem.

Theorem 5.1. *Let p be an even integer. Let $z_1, \ldots, z_N$ be distinct points in $\mathbb{D}$, and consider the zero-set consisting of each of these points with multiplicities $d_1, \ldots, d_N$, respectively. Let G be the canonical divisor for this zero set. Then there are constants c_0 and c_{nj} for $1 \le n \le N$ and $0 \le j \le (p/2)d_n - 1$, such that*

$$G(z)^{p/2} = c_0 + \sum_{n=1}^{N} \sum_{j=0}^{(p/2)d_n-1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}}, \qquad \textit{if } z_n \neq 0 \textit{ for all } n, \textit{ and}$$

$$G(z)^{p/2} = c_0 z^{(p/2)d_1} + \sum_{j=0}^{(p/2)d_1-1} c_{1j} z^j + \sum_{n=2}^{N} \sum_{j=0}^{(p/2)d_n-1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}}, \quad \textit{if } z_1 = 0.$$

Proof. Our goal is to show that $G^{p/2}$ is the kernel for some linear combination of certain derivative-evaluation functionals. Because we know what the kernel of any derivative-evaluation functional is, we will be able to show that G has the form stated in the theorem.

Let $A_n = d_n((p/2)-1)$. For $1 \le n \le N$ and $0 \le j \le A_n - 1$, let h_{nj} be a polynomial such that

$$h_{nj}^{(m)}(z_k) = \begin{cases} 1, & \text{if } m = n \text{ and } k = j \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in A^p$, define

$$\widehat{f}(z) = f(z) - \sum_{n=1}^{N} \sum_{j=0}^{A_n-1} a_{nj} h_{nj}(z)$$

where $a_{nj} = f^{(j)}(z_n)$. Since $\widehat{f}$ has zeros of order $A_n = d_n((p/2)-1)$ at each z_n, the function

$$\widetilde{f} = \frac{1}{G^{(p/2)-1}} \widehat{f}$$

is in A^p.

But then

$$\begin{aligned}
\int_{\mathbb{D}} \overline{G}^{p/2} f \, d\sigma &= \int_{\mathbb{D}} \overline{G(z)}^{p/2} \left(\widehat{f}(z) + \sum_{n=1}^{N} \sum_{j=0}^{A_n-1} a_{nj} h_{nj}(z) \right) d\sigma \\
&= \int_{\mathbb{D}} |G(z)|^{p-1} \overline{\operatorname{sgn} G(z)} \widetilde{f}(z) \, d\sigma + \sum_{n=1}^{N} \sum_{j=0}^{A_n-1} a_{nj} \int_{\mathbb{D}} \overline{G(z)}^{p/2} h_{nj}(z) \, d\sigma \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now, II is a linear combination of the numbers a_{nj} for $1 \le n \le N$ and $0 \le j \le A_n - 1$, so we turn our attention to I. The canonical divisor G is a constant multiple of the function $F \in A^p$ of smallest norm that has zeros of order d_n at each z_j and such that $F^{(m)}(0) = 1$, where m is the order of the zero-set at 0. By Theorem 2.4, $\mathcal{P}(|G|^{p-1} \operatorname{sgn} \overline{G})$ is the kernel for a linear combination of appropriate derivative evaluation functionals at the points $0, z_1, \cdots, z_n$. Thus, we have

$$\begin{aligned}
\int_{\mathbb{D}} |G|^{p-1} \overline{\operatorname{sgn} G} \widetilde{f} \, d\sigma &= \int_{\mathbb{D}} \mathcal{P}(|G|^{p-1} \overline{\operatorname{sgn} G}) \widetilde{f} \, d\sigma \\
&= b_0 \widetilde{f}^{(m)}(0) + \sum_{n=1}^{N} \sum_{j=0}^{d_n-1} b_{nj} \widetilde{f}^{(j)}(z_n),
\end{aligned}$$

for some complex constants b_0 and b_{nj}. Note that $\widetilde{f}(z) = G_n \widehat{f}_n$ where

$$G_n(z) = (z - z_n)^{A_n} \frac{1}{G^{(p/2)-1}(z)}$$

and

$$\widehat{f}_n(z) = (z - z_n)^{-A_n} \widehat{f}(z).$$

Note that

$$\widetilde{f}^{(j)}(z_n) = \sum_{k=0}^{j} \binom{j}{k} \widehat{f}_n^{(k)}(z_n) G_n^{(j-k)}(z_n)$$

and

$$\begin{aligned}\widehat{f}_n^{(k)}(z_n) &= \frac{k!}{(k+A_n)!} \frac{d^{k+A_n}}{dz^{k+A_n}} \widehat{f}(z_n) \\ &= \frac{k!}{(k+A_n)!} \frac{d^{k+A_n}}{dz^{k+A_n}} \left[f(z) - \sum_{n=1}^{N} \sum_{s=0}^{A_n-1} a_{ns} h_{ns}(z) \right]\end{aligned}$$

Thus, $\widetilde{f}^{(j)}(z_n)$ is a linear function of the numbers a_{ns} and the numbers $f^{(k)}(z_n)$ for $0 \le k \le j + A_n$. Recall that $a_{ns} = f^{(s)}(z_n)$.

Also, if $m = 0$, then

$$\widetilde{f}^{(m)}(0) = G(0)^{1-(p/2)} \widehat{f}(0) = G(0)^{1-(p/2)} \left(f(0) - \sum_{n=1}^{N} \sum_{j=0}^{A_n-1} a_{nj} h_{nj}(0) \right),$$

so $\widetilde{f}^{(m)}(0)$ is a linear function of the numbers a_{nj} and $f(0) = f^{(mp/2)}(0)$. If $m \neq 0$, then we may assume $z_1 = 0$ and $m = d_1$, and then by the same reasoning as we used above for $\widetilde{f}^{(j)}(z_n)$, we see that$\widetilde{f}^{(m)}(z_1)$ is a linear function of the numbers a_{nj} and the numbers $f^{(k)}(z_1)$ for $0 \le k \le d_1 + A_1 = d_1 + ((p/2)-1)d_1 = (p/2)d_1 = mp/2$. Thus, term

$$\mathrm{I} = \int_{\mathbb{D}} \overline{G^{(p/2)-1}} \widetilde{f} \, d\sigma$$

is a linear combination of the numbers $f^{(k)}(z_n)$ for $0 \le k \le (d_n - 1) + ((p/2) - 1)d_n = (p/2)d_n - 1$, and the number $f^{(mp/2)}(0)$.

Therefore, both I and II, and thus $\int_{\mathbb{D}} f \overline{G}^{p/2} \, d\sigma$, are linear combinations of the numbers $f^{(k)}(z_n)$ for $0 \le k \le (p/2)d_n - 1$, and the number $f^{(mp/2)}(0)$. And thus, $G^{p/2}$ is the kernel for a derivative-evaluation functional depending on $f^{(j)}(z_n)$ for $1 \le n \le N$ and $0 \le j \le (p/2)d_n - 1$, as well as $f^{mp/2}(0)$. Therefore $G^{p/2}$ has the desired form. □

The previous theorem gave a condition on $G^{p/2}$ that must be satisfied if G is the canonical divisor of a given zero set. The following theorem says that condition, along with a few other more obviously necessary ones, is also sufficient.

Theorem 5.2. *Let p be an even integer. Let $z_1, \ldots, z_N$ be distinct points in $\mathbb{D}$, and consider the zero-set consisting of each of these points with multiplicities $d_1, \ldots, d_N$, respectively. The canonical divisor for this zero set is the unique function G having A^p norm 1 such that $G(0) > 0$ (or $G^{(m)}(0) > 0$ if G is* *required to have a zero of order m at the origin), and such that $G^{p/2}$ has zeros of order $pd_n/2$ at each z_n, and*

$$G(z)^{p/2} = c_0 + \sum_{n=1}^{N} \sum_{j=0}^{(p/2)d_n-1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}} \qquad \text{if } z_n \neq 0 \text{ for all } n \text{ or}$$

$$G(z)^{p/2} = c_0 z^{(p/2)d_1} + \sum_{j=0}^{(p/2)d_1-1} c_{1j} z^j + \sum_{n=2}^{N} \sum_{j=0}^{(p/2)d_n-1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}} \qquad \text{if } z_1 = 0.$$

Proof. By Theorem 5.1 and the definition of the canonical divisor, the stated conditions are necessary for a function to be the canonical divisor. Suppose that G is a function satisfying the stated conditions. We will prove the theorem by applying Theorem 2.4 to $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$.

We first discuss the proof under the assumption that $z_n \neq 0$ for all n. First, as above, $\mathcal{P}\left(\overline{G}^{(p/2)-1}\right) = \overline{G(0)}^{(p/2)-1}$. Now, by Proposition 3.6,

$$\mathcal{P}\left(\frac{1}{(1-\overline{z_n}z)^{j+2}}\,\overline{G(z)}^{(p/2)-1}\right) = \sum_{k=0}^{j-((p/2)-1)d_n} C_{n,j,k} \frac{1}{(1-\overline{z_n}z)^{k+2}},$$

where the constants $C_{n,j,k}$ may depend on G. But if $j \leq (p/2)d_n - 1$, then $j - ((p/2)-1)d_n \leq d_n - 1$. Thus

$$\mathcal{P}\left(G^{p/2}\,\overline{G(z)}^{(p/2)-1}\right) = B_0 + \sum_{n=1}^{N} \sum_{k=0}^{d_n-1} \frac{B_{n,k}}{(1-\overline{z_n}z)^{k+2}},$$

where $B_{n,k} = \sum_{j=k+((p/2)-1)d_n}^{(p/2)d_n-1} c_{nj} C_{n,j,k}$ and $B_0 = c_0\overline{G(0)}^{(p/2)-1}$. By Theorem 2.4, G is a multiple of the canonical divisor. But the conditions that $G^{(m)}(0) > 0$ and $\|G\|_{A^p} = 1$ imply that G is the canonical divisor.

The case where $z_1 = 0$ is similar, but we also use the fact that $\mathcal{P}(z^j\overline{G}^{(p/2)-1})$ is a polynomial of degree at most $j-[(p/2)-1]d_1$, or zero if $j < [(p/2)-1]d_1$. □

From previous work by MacGregor and Stessin [17], a weaker form of Theorem 5.1 is essentially known. In the weaker form of the theorem, one only knows, in the case that no $z_n = 0$, that

$$G(z) = c_0 + \sum_{n=0}^{N} \frac{b_n}{1-\overline{z_n}z} + \sum_{n=1}^{N} \sum_{j=0}^{(p/2)d_n-1} \frac{c_{nj}}{(1-\overline{z_n}z)^{j+2}}$$

for some constants b_n. The case where $z_1 = 0$ is similar. (Although their work also gives a fairly explicit method of finding the canonical divisor, it does not seem to be clear from their results that the b_n will always be zero.) To derive Theorem 5.2 from the weaker form of the theorem, we can use the following proposition, which gives another indication of why the residues of $G^{p/2}$ must all be zero. It basically says that nonzero residues would lead to terms in $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$ that were kernels of functionals of the general form

$$f \mapsto \frac{1}{a}\int_0^a f(z)g(z)\,dz,$$

where g is an analytic function and $a \in \mathbb{D}$. But, as the proposition explains, it would then be impossible for $\mathcal{P}(G^{p/2}\overline{G}^{(p/2)-1})$ to be the kernel of a finite linear combination of derivative-evaluation functionals.

Proposition 5.3. *Let g be analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and suppose g is non-zero on $\partial\mathbb{D}$. Let $a_n \in \mathbb{D}$ and $a_n \neq 0$ for $1 \leq n \leq N$, and assume that $a_n \neq a_j$ for $n \neq j$. Let $b_n \in \mathbb{C}$ for $1 \leq n \leq N$. Then if any of the b_n are nonzero,*

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{g(z)}\right)$$

is not the kernel for a functional that is the finite linear combination of derivative-evaluation functionals.

Note that as is shown in [6] (see also [5], [27], and [7]), the canonical divisor of a finite zero set is analytic in $\overline{\mathbb{D}}$ and non-zero on $\partial\mathbb{D}$. This allows the proposition to be applied to Bergman projections of the form

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{G(z)^{(p/2)-1}}\right).$$

Proof. We know by Proposition 3.7 that

$$\mathcal{P}\left(\sum_{n=1}^{N} \frac{b_n}{1-\overline{a_n}z}\,\overline{g(z)}\right)$$

is the kernel for the functional given by

$$f \mapsto \sum_{n=1}^{N} \frac{b_n}{a_n}\int_0^{a_n} f(z)g(z)\,dz.$$

Suppose this functional were a linear combination of derivative-evaluation functionals, which we will denote by $f \mapsto f^{(k)}(z_j)$, where $1 \leq j \leq J$ and $0 \leq k \leq K$. Let h be a function such that $h = gf$ for some $f \in A^p$. For fixed g, the values $f^{(k)}(z_j)$ for $1 \leq j \leq J$ and $0 \leq k \leq K$ are linear combinations of the values $h^{(k)}(z_j)$, where $1 \leq j \leq J$ and $0 \leq k \leq K + r(z_j)$, and $r(z_j)$ is the order of the zero of g at z_j. Thus the functional defined on the space gA^p by

$$h \mapsto \sum_{n=1}^{N} \frac{b_n}{a_n}\int_0^{a_n} h(z)\,dz$$

must be a linear function of the values $h^{(k)}(z_j)$. By gA^p, we mean the vector space of all functions that may be written as g multiplied by an A^p function. Since g is analytic in $\overline{\mathbb{D}}$ and g is nonzero on $\partial\mathbb{D}$, any polynomial that has all the zeros of g will be in gA^p.

Now for each m there exists a polynomial H_m such that $H_m(a_m) = 1$, but $H_m(a_n) = 0$ for all $n \neq m$, and such that $H_m^{(k)}(z_j) = 0$ for all j and k such that $1 \leq j \leq J$ and $1 \leq k \leq K + r(z_j) + 1$. Also, we may require that H_m' has all the zeros of g, and that $H_m(0) = 0$. Set $h_m = H_m'$. Then h_m shares all the zeros of g, and so it is a multiple of g. Thus

$$\sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} h_m(z)\, dz = 0,$$

since the left side of the above equation is a linear combination of the numbers $h_m^{(k)}(z_j)$ for $1 \leq j \leq J$ and $0 \leq k \leq K + r(z_j)$, and each $h_m^{(k)}(z_j) = 0$. But also, for each m such that $1 \leq m \leq N$, we have

$$\sum_{n=1}^{N} \frac{b_n}{a_n} \int_0^{a_n} h_m(z)\, dz = \sum_{n=1}^{N} \frac{b_n}{a_n} H_m(a_n) = \frac{b_m}{a_m},$$

so each $b_m = 0$. □

Example 5.4. Suppose we are given distinct points $z_1, z_2, \ldots, z_N \in \mathbb{D}\backslash\{0\}$. Let $p = 2M$, where M is a positive integer. Suppose we wish to find the canonical divisor in A^p for the given set of points. From the theorem, we know that

$$G(z)^M = c_0 + \sum_{n=1}^{N} \sum_{m=0}^{M-1} \frac{c_{nm}}{(1 - \overline{z_n} z)^{m+2}}.$$

Then we have for $0 \leq k \leq M - 1$ and $1 \leq n \leq N$ that

$$0 = \frac{d^k}{dz^k} \left(G(z)^M \right) |_{z=z_j} = \frac{d^k}{dz^k} c_0 + \sum_{n=1}^{N} \sum_{m=0}^{M-1} \frac{\overline{z_n}^k (m+1+k)!/(m+1)!}{(1 - \overline{z_n} z_j)^{m+2+k}} c_{nm}.$$

This gives a system of NM equations with $NM + 1$ unknowns. Because of the uniqueness of the canonical divisor, there will be a unique solution to these equations with $c_0 > 0$ and such that $\|G\|_{A^p} = 1$.

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