EXTREMAL PROBLEMS IN BERGMAN SPACES AND AN EXTENSION OF RYABYKH'S THEOREM

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Abstract. We study linear extremal problems in the Bergman space A^p of the unit disc for p an even integer. Given a functional on the dual space of A^p with representing kernel $k \in A^q$, where $1/p + 1/q = 1$, we show that if the Taylor coefficients of k are sufficiently small, then the extremal function $F \in H^\infty$. We also show that if $q \le q_1 < \infty$, then $F \in H^{(p-1)q_1}$ if and only if $k \in H^{q_1}$.

An analytic function f in the unit disc $\mathbb{D}$ is said to belong to the Bergman space A^p if

$$\|f\|_{A^p} = \left\{ \int_{\mathbb{D}} |f(z)|^p d\sigma(z) \right\}^{1/p} < \infty.$$

Here σ denotes normalized area measure, so that $\sigma(\mathbb{D}) = 1$. For $1 < p < \infty$, each functional $\phi \in (A^p)^*$ has a unique representation

$$\phi(f) = \int_{\mathbb{D}} f\overline{k}\, d\sigma,$$

for some $k \in A^q$, where $q = p/(p-1)$ is the conjugate index. The function k is called the kernel of the functional ϕ.

In this paper we study the extremal problem of maximizing $\operatorname{Re}\phi(f)$ among all functions $f \in A^p$ of unit norm. If $1 < p < \infty$, then an extremal function always exists and is unique. However, to find it explicitly is in general a difficult problem, and few explicit solutions are known. Here we consider the problem of determining whether the kernel being "well-behaved" implies that the extremal function is also "well-behaved." A known result in this direction is Ryabykh's theorem, which states that if the kernel is actually in the Hardy space H^q, then the extremal function must be in the Hardy space H^p. In [4], we gave a proof of Ryabykh's theorem based on general properties of extremal functions in uniformly convex spaces.

In this paper, we obtain a sharper version of Ryabykh's theorem in the case where p is an even integer. Our results are:

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- For $q \le q_1 < \infty$, the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$.
- If the Taylor coefficients of k are "small enough," then $F \in H^\infty$.
- The map sending a kernel $k \in H^q$ to its extremal function $F \in A^p$ is a continuous map from $H^q \setminus 0$ into H^p.

Our proofs rely heavily on Littlewood-Paley theory, and seem to require that p be an even integer. It is an open problem whether the results hold without this assumption.

1. Extremal Problems and Ryabykh's Theorem

We begin with some notation. If f is an analytic function, $S_n f$ denotes its n^{th} Taylor polynomial at the origin. Lebesgue area measure is denoted by dA, and $d\sigma$ denotes normalized area measure.

If h is a measurable function in the unit disc, the principal value of its integral is

$$\text{p. v.} \int_{\mathbb{D}} h\, dA = \lim_{r \to 1} \int_{r\mathbb{D}} h\, dA,$$

if the limit exists.

We now recall some basic facts about Hardy and Bergman spaces. For proofs and further information, see [2] and [3]. Suppose that f is analytic in the unit disc. For $0 < p < \infty$ and $0 < r < 1$, the integral mean of f is

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

If $p = \infty$, we write

$$M_\infty(f, r) = \max_{0 \le \theta < 2\pi} |f(re^{i\theta})|.$$

For fixed f and p, the integral means are increasing functions of r. If $M_p(f, r)$ is bounded we say that f is in the Hardy space H^p. For any function f in H^p, the radial limit $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exists for almost every θ. An H^p function is uniquely determined by the values of its boundary function on any set of positive measure. The space H^p is a Banach space with norm

$$\|f\|_{H^p} = \sup_r M_p(f, r) = \|f(e^{i\theta})\|_{L^p}.$$

It is useful to regard H^p as a subspace of $L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle. For $0 < p < \infty$, if $f \in H^p$, then $f(re^{i\theta})$ converges to $f(e^{i\theta})$ in L^p norm as $r \to 1$.

For $1 < p < \infty$, the dual space $(H^p)^*$ is isomorphic to H^q, where $1/p + 1/q = 1$, with an element $k \in H^q$ representing the functional ϕ defined by

$$\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{k(e^{i\theta})}\, d\theta.$$

This isomorphism is not an isometry unless $p = 2$, but it is true that $\|\phi\|_{(H^p)^*} \leq \|k\|_{H^q} \leq C\|\phi\|_{(H^p)^*}$ for some constant C depending only on p. If $f \in H^p$ for $1 < p < \infty$, then $S_n f \to f$ in H^p as $n \to \infty$. The Szegő projection S maps each function $f \in L^1(\mathbb{T})$ into a function analytic in $\mathbb{D}$ defined by

$$Sf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z} dt.$$

It leaves H^1 functions fixed and maps L^p boundedly into L^p for $1 < p < \infty$. If $f \in L^p$ for $1 < p < \infty$ and $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then $Sf(z) = \sum_{n=0}^{\infty} a_n z^n$.

For $1 < p < \infty$, the dual of the Bergman space A^p is isomorphic to A^q, where $1/p + 1/q = 1$, and $k \in A^q$ represents the functional defined by $\phi(f) = \int_{\mathbb{D}} f(z)\overline{k(z)}\, d\sigma(z)$. Note that this isomorphism is actually conjugate-linear. It is not an isometry unless $p = 2$, but if the functional $\phi \in (A^p)^*$ is represented by the function $k \in A^q$, then

$$\|\phi\|_{(A^p)^*} \leq \|k\|_{A^q} \leq C_p\|\phi\|_{(A^p)^*} \tag{1.1}$$

where C_p is a constant depending only on p. We remark that $H^p \subset A^p$, and in fact $\|f\|_{A^p} \leq \|f\|_{H^p}$. If $f \in A^p$ for $1 < p < \infty$, then $S_n f \to f$ in A^p as $n \to \infty$.

In this paper the only Bergman spaces we consider are those with $1 < p < \infty$. For a given linear functional $\phi \in (A^p)^*$ such that $\phi \neq 0$, we investigate the extremal problem of finding a function $F \in A^p$ with norm $\|F\|_{A^p} = 1$ for which

$$\operatorname{Re} \phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re} \phi(g) = \|\phi\|. \tag{1.2}$$

Such a function F is called an extremal function, and we say that F is an extremal function for a function $k \in A^q$ if F solves problem (1.2) for the functional ϕ with kernel k. This problem has been studied by Vukotić [10], Khavinson and Stessin [7], and Ferguson [4], among others. Note that for $p = 2$, the extremal function is $F = k/\|k\|_{A^2}$.

A closely related problem is that of finding $f \in A^p$ such that $\phi(f) = 1$ and

$$\|f\|_{A^p} = \inf_{\phi(g)=1} \|g\|_{A^p}. \tag{1.3}$$

If F solves the problem (1.2), then $\frac{F}{\phi(F)}$ solves the problem (1.3), and if f solves (1.3), then $\frac{f}{\|f\|}$ solves (1.2). When discussing either of these problems, we always assume that ϕ is not the zero functional; in other words, that k is not identically 0.

The problems (1.2) and (1.3) each have a unique solution when $1 < p < \infty$ (see [4], Theorem 1.4). Also, for every function $f \in A^p$ such that f is not identically 0, there is a unique $k \in A^q$ such that f solves problem (1.3) for k (see [4], Theorem 3.3). This implies that for each $F \in A^p$ with $\|F\|_{A^p} = 1$, there is some nonzero k such that F solves problem (1.2) for k. Furthermore, any two such kernels k are positive multiples of each other.

The Cauchy-Green theorem is an important tool in this paper.

Cauchy-Green Theorem. *If Ω is a region in the plane with piecewise smooth boundary and $f \in C^1(\overline{\Omega})$, then*

$$\frac{1}{2i}\int_{\partial\Omega} f(z)\,dz = \int_{\Omega} \frac{\partial}{\partial \overline{z}} f(z)\,dA(z),$$

where $\partial\Omega$ denotes the boundary of Ω.

The next result is an important characterization of extremal functions in A^p for $1 < p < \infty$ (see [9], p.55).

Theorem A. *Let $1 < p < \infty$ and let $\phi \in (A^p)^*$. A function $F \in A^p$ with $\|F\|_{A^p} = 1$ satisfies*

$$\operatorname{Re}\phi(F) = \sup_{\|g\|_{A^p}=1} \operatorname{Re}\phi(g) = \|\phi\|$$

if and only if

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\,d\sigma = 0$$

for all $h \in A^p$ with $\phi(h) = 0$. If F satisfies the above conditions, then

$$\int_{\mathbb{D}} h|F|^{p-1}\overline{\operatorname{sgn} F}\,d\sigma = \frac{\phi(h)}{\|\phi\|}$$

for all $h \in A^p$.

Ryabykh's theorem relates extremal problems in Bergman spaces to Hardy spaces. It says that if the kernel for a linear functional is not only in A^q but also in H^q, then the extremal function is not only in A^p but in H^p as well.

Ryabykh's Theorem. *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Suppose that $\phi \in (A^p)^*$ and $\phi(f) = \int_{\mathbb{D}} f\overline{k}\,d\sigma$ for some $k \in H^q$. Then the solution F to the extremal problem (1.2) belongs to H^p and satisfies*

$$\|F\|_{H^p} \le \left\{ \left[\max(p-1,1)\right] \frac{C_p\|k\|_{H^q}}{\|k\|_{A^q}} \right\}^{1/(p-1)}, \tag{1.4}$$

where C_p is the constant in (1.1).

Ryabykh[8] proved that $F \in H^p$. The bound (1.4) was proved in [4], by a variant of Ryabykh's proof.

As a corollary Ryabykh's theorem implies that the solution to the problem (1.3) is in H^p as well. Note that the constant $C_p \to \infty$ as $p \to 1$ or $p \to \infty$.

To obtain our results, including a generalization of Ryabykh's theorem, we will need the following technical lemmas. Their proofs, which involve Littlewood-Paley theory, are deferred to the end of the paper.

Lemma 1.1. *Let p be an even integer. Let $f \in H^p$ and let h be a polynomial. Then*

$$\text{p. v.} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} f' h \, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} |f|^{p-1} \overline{\operatorname{sgn} f} (S_n f)' h \, d\sigma.$$

Lemma 1.2. *Suppose that $1 < p_1 < \infty$ and $1 < p_2, p_3 \le \infty$, and also that*

$$1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Let $f_1 \in H^{p_1}$, $f_2 \in H^{p_2}$, and $f_3 \in H^{p_3}$. Then

$$\left| \text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3' \, d\sigma \right| \le C \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}}$$

where C depends only on p_1 and p_2. (Implicit is the claim that the principal value exists.) Moreover, if $p_3 < \infty$, then

$$\text{p. v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3' \, d\sigma = \lim_{n\to\infty} \int_{\mathbb{D}} \overline{f_1} f_2 (S_n f_3)' \, d\sigma.$$

2. The Norm-Equality

Let p be an even integer and let q be its conjugate exponent. Let $k \in H^q$ and let F be the extremal function for k over A^p. We will denote by ϕ the functional associated with k. Let F_n be the extremal function for k when the extremal problem is posed over P_n, the space of polynomials of degree at most n. Also, let

$$K(z) = \frac{1}{z} \int_0^z k(\zeta) \, d\zeta, \tag{2.1}$$

so that $(zK)' = k$. During proof of Ryabykh's theorem in [4], an important step is to show that

$$\frac{1}{2\pi} \int_0^{2\pi} |F_n(e^{i\theta})|^p d\theta = \frac{1}{2\pi \|\phi_{|P_n}\|} \int_0^{2\pi} F_n \left[\left(\frac{p}{2}\right) \overline{k} + \left(1 - \frac{p}{2}\right) \overline{K} \right] d\theta,$$

(see [4], p. 2652). We will now derive a similar result for F:

Theorem 2.1. *Let p be an even integer, let $k \in H^q$, and let $F \in A^p$ be the extremal function for k. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta = \frac{1}{2\pi \|\phi\|} \int_0^{2\pi} F \left[\left(\frac{p}{2}\right) h\overline{k} + \left(1 - \frac{p}{2}\right) (zh)' \overline{K} \right] d\theta,$$

for every polynomial h.

Proof. Since Ryabykh's theorem says that $F \in H^p$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta}) d\theta = \lim_{r\to 1} \frac{i}{2\pi} \int_{\partial(r\mathbb{D})} |F(z)|^p h(z) z \, d\overline{z},$$

where h is any polynomial. Apply the Cauchy-Green theorem to transform the right-hand side into

$$\text{p. v.}\,\frac{1}{\pi}\int_{\mathbb{D}}\left((zh)'F+\frac{p}{2}zhF'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\,dA(z).$$

Invoking Lemma 1.1 with zh in place of h shows that this limit equals

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{\mathbb{D}}\left((zh)'F+\frac{p}{2}zh(S_nF)'\right)|F|^{p-1}\overline{\operatorname{sgn} F}\,dA(z).$$

Since $(zh)'F+\frac{p}{2}zh(S_nF)'$ is in A^p, we may apply Theorem A to reduce the last expression to

$$\lim_{n\to\infty}\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}}\left((zh)'F+\frac{p}{2}zh(S_nF)'\right)\overline{k}\,dA(z). \tag{2.2}$$

Recall that we have defined $K(z)=\frac{1}{z}\int_0^z k(\zeta)\,d\zeta$. To prepare for a reverse application of the Cauchy-Green theorem, we rewrite the integral in (2.2) as

$$\frac{1}{\pi\|\phi\|}\int_{\mathbb{D}}\left[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\}+\frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\}\right.$$
$$\left.-\frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\right]dA(z).$$

Now this equals

$$\lim_{r\to1}\frac{1}{\pi\|\phi\|}\int_{r\mathbb{D}}\left[\frac{\partial}{\partial\overline{z}}\left\{(zh)'F\overline{zK}\right\}+\frac{p}{2}\frac{\partial}{\partial z}\left\{zhS_n(F)\overline{k}\right\}\right.$$
$$\left.-\frac{p}{2}\frac{\partial}{\partial\overline{z}}\left\{(zh)'S_n(F)\overline{zK}\right\}\right]dA(z).$$

We apply the Cauchy-Green theorem to show that this equals

$$\lim_{r\to1}\left[\frac{1}{2\pi i\|\phi\|}\int_{\partial(r\mathbb{D})}(zh)'F\overline{zK}\,dz+\frac{ip}{4\pi\|\phi\|}\int_{\partial(r\mathbb{D})}zhS_n(F)\overline{k}\,d\overline{z}\right.$$
$$\left.-\frac{p}{4\pi i\|\phi\|}\int_{\partial(r\mathbb{D})}(zh)'S_n(F)\overline{zK}\,dz\right].$$

Since F is in H^p and both k and K are in H^q, the above limit equals

$$\frac{1}{2\pi i\|\phi\|}\int_{\partial\mathbb{D}}(zh)'F\overline{zK}\,dz+\frac{ip}{4\pi\|\phi\|}\int_{\partial\mathbb{D}}zhS_n(F)\overline{k}\,d\overline{z}$$
$$-\frac{p}{4\pi i\|\phi\|}\int_{\partial\mathbb{D}}(zh)'S_n(F)\overline{zK}\,dz$$
$$=\frac{1}{2\pi\|\phi\|}\int_0^{2\pi}(zh)'F\overline{K}+S_n(F)\left(\frac{p}{2}h\overline{k}-\frac{p}{2}(zh)'\overline{K}\right)d\theta.$$

We let $n\to\infty$ in the above expression to reach the desired conclusion. □

Taking $h = 1$, we have the following corollary, which we call the "norm-equality".

Corollary 2.2. (The Norm-Equality). *Let p be an even integer, let $k \in H^q$, and let F be the extremal function for k. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p d\theta = \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F\left[\left(\frac{p}{2}\right)\overline{k} + \left(1-\frac{p}{2}\right)\overline{K}\right] d\theta.$$

The norm-equality is useful mainly because it yields the following theorem.

Theorem 2.3. *Let p be an even integer. Let $\{k_n\}$ be a sequence of H^q functions, and let $k_n \to k$ in H^q. Let F_n be the A^p extremal function for k_n and let F be the A^p extremal function for k. Then $F_n \to F$ in H^p.*

Note that Ryabykh's theorem shows that each $F_n \in H^p$, and that $F \in H^p$. But because the operator taking a kernel to its extremal function is not linear, one cannot apply the closed graph theorem to conclude that $F_n \to F$.

To prove Theorem 2.3 we will use the following lemma involving the notion of uniform convexity. A Banach space X is called *uniformly convex* if for each $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$\left\|\tfrac{1}{2}(x+y)\right\| > 1-\delta \qquad \text{implies} \qquad \|x-y\| < \epsilon.$$

An equivalent definition is that if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|x_n\| = \|y_n\| = 1$ for all n and $\|x_n + y_n\| \to 2$ then $\|x_n - y_n\| \to 0$. This concept was introduced by Clarkson in [1]. See also [4], where it is applied to extremal problems. To apply the lemma, we use the fact that the space H^p is uniformly convex for $1 < p < \infty$. By $x_n \rightharpoonup x$, we mean that x_n approaches x weakly.

Lemma 2.4. *Suppose that X is a uniformly convex Banach space, that $x \in X$, and that $\{x_n\}$ is a sequence of elements of X. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ in X.*

This lemma is known. For example, it is contained in Exercise 15.17 in [6].

Proof of Theorem. We will first show that $F_n \rightharpoonup F$ in H^p (that is, F_n converges to F weakly in H^p). Next we will use this fact and the norm-equality to show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$. By the lemma, it will then follow that $F_n \to F$ in H^p.

To prove that $F_n \rightharpoonup F$ in H^p, note that Ryabykh's theorem says that $\|F_n\|_{H^p} \le C(\|k_n\|_{H^q}/\|k_n\|_{A^q})^{1/(p-1)}$. Let $\alpha = \inf_n \|k_n\|_{A^q}$ and $\beta = \sup_n \|k_n\|_{H^q}$. Here $\alpha > 0$ because by assumption none of the k_n are identically zero, and they approach k, which is not identically 0. Therefore $\|F_n\|_{H^p} \le C(\beta/\alpha)^{1/(p-1)}$, and the sequence $\{F_n\}$ is bounded in H^p norm.

Now, suppose that $F_n \not\rightharpoonup F$. Then there is some $\psi \in (H^p)^*$ such that $\psi(F_n) \not\rightharpoonup \psi(F)$. This implies $|\psi(F_{n_j}) - \psi(F)| \ge \epsilon$ for some $\epsilon > 0$ and some subsequence $\{F_{n_j}\}$. But since the sequence $\{F_n\}$ is bounded in H^p norm, the Banach-Alaoglu theorem implies that some subsequence of $\{F_{n_j}\}$, which we will also denote by $\{F_{n_j}\}$, converges weakly in H^p to some function $\widetilde{F}$. Then $|\psi(\widetilde{F}) - \psi(F)| \geq \epsilon$. Now $k_n \to k$ in A^q, and it is proved in [4] that this implies $F_n \to F$ in A^p, which implies $F_n(z) \to F(z)$ for all $z \in \mathbb{D}$. Since point evaluation is a bounded linear functional on H^p, we have that $F_{n_j}(z) \to \widetilde{F}(z)$ for all $z \in \mathbb{D}$, which means that $\widetilde{F}(z) = F(z)$ for all $z \in \mathbb{D}$. But this contradicts the assumption that $\psi(\widetilde{F}) \neq \psi(F)$. Hence $F_n \rightharpoonup F$.

Let ϕ_n be the functional with kernel k_n, and let ϕ be the functional with kernel k. To show that $\|F_n\|_{H^p} \to \|F\|_{H^p}$, recall that the norm-equality says

$$\frac{1}{2\pi}\int_0^{2\pi} |F_n(e^{i\theta})|^p d\theta = \frac{1}{2\pi\|\phi_n\|}\int_0^{2\pi} F_n\left[\left(\frac{p}{2}\right)\overline{k_n} + \left(1-\frac{p}{2}\right)\overline{K_n}\right] d\theta.$$

But, if h is any function analytic in $\mathbb{D}$ and $H(z) = (1/z)\int_0^z h(\zeta)d\zeta$, it can be shown that $\|H\|_{H^q} \leq \|h\|_{H^q}$ (see [4], proof of Theorem 4.2). Since $k_n \to k$ in H^q, it follows that $K_n \to K$ in H^q. Also, $k_n \to k$ in A^p implies that $\|\phi_n\| \to \|\phi\|$. In addition, $\|F_n\|_{H^p} \leq C$ for some constant C, and $F_n \rightharpoonup F$, so the right-hand side of the above equation approaches

$$\frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F\left[\left(\frac{p}{2}\right)\overline{k} + \left(1-\frac{p}{2}\right)\overline{K}\right] d\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p d\theta.$$

In other words, $\|F_n\|_{H^p} \to \|F\|_{H^p}$, and so by Lemma 2.4 we conclude that $F_n \to F$ in H^p. □

3. Fourier Coefficients of $|F|^p$

Theorem 2.1 can also be used to gain information about the Fourier coefficients of $|F|^p$, where F is the extremal function. In particular, it leads to a criterion for F to be in L^∞ in terms of the Taylor coefficients of the kernel k.

Theorem 3.1. *Let p be an even integer. Let $k \in H^q$, let F be the A^p extremal function for k, and define K by equation* (2.1)*. Then for any integer $m \geq 0$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p e^{im\theta} d\theta = \frac{1}{2\pi\|\phi\|}\int_0^{2\pi} F e^{im\theta}\left[\left(\frac{p}{2}\right)\overline{k} + \left(1-\frac{p}{2}\right)(m+1)\overline{K}\right] d\theta.$$

Proof. Take $h(e^{i\theta}) = e^{im\theta}$ in Theorem 2.1. □

This last formula can be applied to obtain estimates on the size of the Fourier coefficients of $|F|^p$.

Theorem 3.2. *Let p be an even integer. Let $k \in A^q$, and let F be the A^p extremal function for k. Let*

$$b_m = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p e^{-im\theta} d\theta,$$

and let

$$k(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then, for each $m \geq 0$,

$$|b_m| = |b_{-m}| \leq \frac{p}{2\|\phi\|} \|F\|_{H^2} \left[\sum_{n=m}^{\infty} |c_n|^2 \right]^{1/2}.$$

Proof. The theorem is trivially true if $k \notin H^2$, so we may assume that $k \in A^2 \subset A^q$. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $F \in H^p$, and $p \geq 2$, we have $F \in H^2$. Now, using Theorem 3.1, we find that

$$\begin{aligned}
b_{-m} &= \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p e^{im\theta} d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} (Fe^{im\theta}) \left[\left(\frac{p}{2}\right) \overline{k} + \left(1 - \frac{p}{2}\right)(m+1)\overline{K} \right] d\theta \\
&= \frac{1}{2\pi\|\phi\|} \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n e^{i(n+m)\theta} \right] \left[\sum_{j=0}^{\infty} \left(\left(\frac{p}{2}\right) \overline{c_j} + \frac{m+1}{j+1} \left(1 - \frac{p}{2}\right) \overline{c_j} \right) e^{-ij\theta} \right] d\theta \\
&= \frac{1}{\|\phi\|} \left| \sum_{n=0}^{\infty} a_n \left(\left(\frac{p}{2}\right) \overline{c_{n+m}} + \frac{m+1}{n+m+1} \left(1 - \frac{p}{2}\right) \overline{c_{n+m}} \right) \right|.
\end{aligned}$$

The Cauchy-Schwarz inequality now gives

$$\begin{aligned}
|b_{-m}| &\leq \frac{1}{\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} \left[\sum_{n=m}^{\infty} \left| \left(\frac{p}{2}\right) \overline{c_n} + \frac{m+1}{n+1} \left(1 - \frac{p}{2}\right) \overline{c_n} \right|^2 \right]^{1/2} \\
&\leq \frac{p}{2\|\phi\|} \left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} \left[\sum_{n=m}^{\infty} |c_n|^2 \right]^{1/2}.
\end{aligned}$$

Since

$$\left[\sum_{n=0}^{\infty} |a_n|^2 \right]^{1/2} = \|F\|_{H^2}$$

the theorem follows. □

The estimate in Theorem 3.2 can be used to obtain information about the size of $|F|^p$ and F, as in the following corollary.

Corollary 3.3. *If* $c_n = O(n^{-\alpha})$ *for some* $\alpha > 3/2$*, then* $F \in H^\infty$.

Proof. First observe that

$$\sum_{n=m}^{\infty} (n^{-\alpha})^2 \leq \int_{m-1}^{\infty} x^{-2\alpha} dx = \frac{(m-1)^{1-2\alpha}}{2\alpha - 1}.$$

By hypothesis it follows that

$$\left[\sum_{n=m}^{\infty} |c_n|^2\right]^{1/2} = O(m^{(1-2\alpha)/2}).$$

Thus, Theorem 3.2 shows that $b_m = O(m^{(1-2\alpha)/2})$. Therefore $\{b_m\} \in \ell^1$ if $\alpha > 3/2$. But $\{b_m\} \in \ell^1$ implies $|F|^p \in L^\infty$, which implies $F \in H^\infty$. □

In fact, $\{b_m\} \in \ell^1$ implies that $|F|^p$ is continuous in $\overline{\mathbb{D}}$, but this does not necessarily mean F will be continuous in $\overline{\mathbb{D}}$. There is a result similar to Corollary 3.3 in [7], where the authors show that if the kernel k is a polynomial, or even a rational function with no poles in $\overline{\mathbb{D}}$, then F is Hölder continuous in $\overline{\mathbb{D}}$. Their technique relies on deep regularity results for partial differential equations. Our result only shows that $F \in H^\infty$, but it applies to a broader class of kernels.

4. Relations Between the Size of the Kernel and Extremal Function

In this section we show that if p is an even integer and $q \le q_1 < \infty$, then the extremal function $F \in H^{(p-1)q_1}$ if and only if the kernel $k \in H^{q_1}$. For $q_1 = q$ the statement reduces to Ryabykh's theorem and its previously unknown converse. The following theorem is crucial to the proof.

Theorem 4.1. *Let p be an even integer and let $q = p/(p-1)$ be its conjugate exponent. Let $F \in A^p$ be the extremal function corresponding to the kernel $k \in A^q$. Suppose that $k \in H^{q_1}$ for some q_1 with $q \le q_1 < \infty$, and that $F \in H^{p_1}$, for some p_1 with $p \le p_1 < \infty$. Define p_2 by*

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

If $p_2 < \infty$, then for every trigonometric polynomial h we have

$$\left|\int_0^{2\pi} |F|^p h(e^{i\theta})\, d\theta\right| \le C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \|F\|_{H^{p_1}} \|h\|_{L^{p_2}},$$

where C is some constant depending only on p, p_1, and q_1.

The excluded case $p_2 = \infty$ occurs if and only if $q = q_1$ and $p = p_1$. The theorem is then a trivial consequence of Ryabykh's theorem.

Proof of Theorem. First let h be an analytic polynomial. In the proof of Theorem 2.1, we showed that

(4.1)
$$\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^p h(e^{i\theta})\, d\theta = \lim_{n\to\infty} \frac{1}{\pi\|\phi\|} \int_{\mathbb{D}} \Big((hz)'F + \frac{p}{2} hz(S_nF)'\Big)\, \overline{k}\, dA(z).$$

An application of Lemma 1.2 gives

$$\lim_{n\to\infty}\int_{\mathbb{D}} hz(S_nF)'\overline{k}\,dA = \text{p. v.}\int_{\mathbb{D}} hzF'\overline{k}\,dA,$$

so that the right-hand side of equation (4.1) becomes

$$\frac{1}{\pi\|\phi\|}\,\text{p. v.}\int_{\mathbb{D}}\Big((hz)'F+\frac{p}{2}hzF'\Big)\,\overline{k}\,dA(z).$$

Apply Lemma 1.2 separately to the two parts of the integral to conclude that its absolute value is bounded by

$$C\frac{1}{\|\phi\|}\|k\|_{H^{q_1}}\|f\|_{H^{p_1}}\|h\|_{H^{p_2}},$$

where C is a constant depending only on p_1 and q_1. Since

$$\frac{1}{\|\phi\|}\le\frac{C_p}{\|k\|_{A^q}}$$

by equation (1.1), this gives the desired result for the special case where h is an analytic polynomial.

Now let h be an arbitrary trigonometric polynomial. Then $h = h_1+\overline{h_2}$, where h_1 and h_2 are analytic polynomials, and $h_2(0)=0$. Note that the Szegő projection S is bounded from L^{p_2} into H^{p_2} because $1<p_2<\infty$. Thus,

$$\|h_1\|_{H^{p_2}}=\|S(h)\|_{H^{p_2}}\le C\|h\|_{L^{p_2}}.$$

Also,

$$\|h_2\|_{H^{p_2}}=\|zS(e^{-i\theta}\overline{h})\|_{H^{p_2}}=\|S(e^{-i\theta}\overline{h})\|_{H^{p_2}}\le C\|e^{-i\theta}\overline{h}\|_{L^{p_2}}=C\|h\|_{L^{p_2}},$$

and so

$$\|h_1\|_{H^{p_2}}+\|h_2\|_{H^{p_2}}\le C\|h\|_{L^{p_2}}.$$

Therefore, by what we have already shown,

$$\begin{aligned}\left|\int_0^{2\pi}|f(e^{i\theta})|^p h(e^{i\theta})d\theta\right| &= \left|\int_0^{2\pi}|f(e^{i\theta})|^p(h_1(e^{i\theta})+\overline{h_2(e^{i\theta})})d\theta\right|\\ &\le\left|\int_0^{2\pi}|f|^p h_1\,d\theta\right|+\left|\overline{\int_0^{2\pi}|f|^p h_2\,d\theta}\right|\\ &\le C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|f\|_{H^{p_1}}(\|h_1\|_{H^{p_2}}+\|h_2\|_{H^{p_2}})\\ &\le C\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\|f\|_{H^{p_1}}\|h\|_{L^{p_2}}.\end{aligned}$$

□

For a given q_1, we will apply the theorem just proved with p_1 chosen as $p_1 = pp_2'$, where p_2' is the conjugate exponent to p_2. This will allow us to bound the H^{p_1} norm of f solely in terms of $\|\phi\|$ and $\|k\|_{H^{q_1}}$.

Theorem 4.2. *Let p be an even integer, and let q be its conjugate exponent. Let $F \in A^p$ be the extremal function for a kernel $k \in A^q$. If, for q_1 such that $q \le q_1 < \infty$, the kernel $k \in H^{q_1}$, then $F \in H^{p_1}$ for $p_1 = (p-1)q_1$. In fact,*

$$\|F\|_{H^{p_1}} \le C \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)},$$

where C depends only on p and q_1.

Proof. The case $q_1 = q$ is Ryabykh's theorem, so we assume $q_1 > q$. Set $p_1 = (p-1)q_1$. Then $p_1 > p = (p-1)q$. Choose p_2 so that

$$\frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

This implies that $p_2 = p_1/(p_1 - p)$, and so its conjugate exponent $p_2' = p_1/p$. Note that $1 < p_2 < \infty$. Let F_n denote the extremal function corresponding to the kernel $S_n k$, which does not vanish identically if n is chosen sufficiently large. Since $S_n k$ is a polynomial, F_n is in H^∞ (and thus $F_n \in H^{p_1}$) by Corollary 3.3. Hence for any trigonometric polynomial h, Theorem 4.1 yields

$$\left| \frac{1}{2\pi} \int_0^{2\pi} |F_n|^p h(e^{i\theta}) d\theta \right| \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}} \|h\|_{L^{p_2}}.$$

Since the trigonometric polynomials are dense in $L^{p_2}(\partial\mathbb{D})$, taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{p_2}} \le 1$ gives

$$\| |F_n|^p \|_{L^{p_2'}} \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}},$$

which implies

$$\begin{aligned} \|F_n\|_{H^{p_1}}^p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} (|F_n(e^{i\theta})|^p)^{p_2'} d\theta \right\}^{1/p_2'} &= \| |F_n|^p \|_{L^{p_2'}} \\ &\le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}} \|F_n\|_{H^{p_1}}, \end{aligned}$$

since $pp_2' = p_1$. Because $\|F_n\|_{H^{p_1}} < \infty$, we may divide both sides of the inequality by $\|F_n\|_{H^{p_1}}$ to obtain

$$\|F_n\|_{H^{p_1}}^{p-1} \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}},$$

where C depends only on p and q_1. In other words,

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F_n(re^{i\theta})|^{p_1} d\theta \right)^{(p-1)/p_1} \le C \frac{\|S_n k\|_{H^{q_1}}}{\|S_n k\|_{A^q}}$$

for all $r < 1$ and for all n sufficiently large. Note that $S_n k \to k$ in H^{q_1} and in A^q. Since $S_n k \to k$ in A^q, Theorem 3.1 in [4] says that $F_n \to F$ in A^p, and thus $F_n \to F$ uniformly on compact subsets of $\mathbb{D}$. Thus, letting $n \to \infty$ in the last inequality gives

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^{p_1} d\theta \right)^{(p-1)/p_1} \leq C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}$$

for all $r < 1$. In other words,

$$\|F\|_{H^{p_1}} \leq \left(C \frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \right)^{1/(p-1)}. \qquad \square$$

Recall from Section 1 that a function $F \in A^p$ with unit norm has a corresponding kernel $k \in A^q$ such that F is the extremal function for k, and this kernel is uniquely determined up to a positive multiple. Theorem 4.2 says that if p is an even integer and a kernel k belongs not only to the Bergman space A^q but also to the Hardy space H^{q_1} for some q_1 where $q \leq q_1 < \infty$, then the A^p extremal function F associated with it is actually in H^{p_1} for $p_1 = (p-1)q_1 \geq p$. It is natural to ask whether the converse is true. In other words, if $F \in H^{p_1}$ for some p_1 with $p \leq p_1 < \infty$, must it follow that the corresponding kernel belongs to H^{q_1}? The following theorem says that this is indeed the case.

Theorem 4.3. *Suppose p is an even integer and let q be its conjugate exponent. Let $F \in A^p$ with $\|F\|_{A^p} = 1$, and let k be a kernel such that F is the extremal function for k. If $F \in H^{p_1}$ for some p_1 with $p \leq p_1 < \infty$, then $k \in H^{q_1}$ for $q_1 = p_1/(p-1)$, and*

$$\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}} \leq C \|F\|_{H^{p_1}}^{p-1},$$

where C is a constant depending only on p and p_1.

Proof. Let h be a polynomial and let ϕ be the functional in $(A^p)^*$ corresponding to k. Then by Theorem A,

$$\begin{aligned} \frac{1}{\|\phi\|} \int_{\mathbb{D}} \overline{k(z)} (zh(z))' d\sigma &= \int_{\mathbb{D}} |F(z)|^{p-1} \operatorname{sgn}(\overline{F(z)}) (zh(z))' d\sigma \\ &= \int_{\mathbb{D}} \overline{F^{p/2}} F^{(p/2)-1} (zh(z))' d\sigma. \end{aligned}$$

By hypothesis, $F^{p/2} \in H^{(2p_1)/p}$ and $F^{(p/2)-1} \in H^{2p_1/(p-2)}$. A simple calculation shows that

$$\frac{1}{q_1'} = \frac{q_1 - 1}{q_1} = \frac{p_1 - p + 1}{p_1}$$

and thus

$$\frac{p}{2p_1} + \frac{p-2}{2p_1} + \frac{1}{q_1'} = 1.$$

Now we will apply the first part of Lemma 1.2 with $f_1 = F^{p/2}$ and $f_2 = F^{(p/2)-1}$ and $f_3 = zh$, and with $2p_1/p$ in place of p_1, and $2p_1/(p-2)$ in place of p_2, and q_1' in place of p_3. Note that this is permitted since $1 < 2p_1/p < \infty$, and $1 < q_1' < \infty$, and $1 < 2p_1/(p-2) \leq \infty$. (In fact, we even know that $2p_1/(p-2) < \infty$ unless $p = 2$, which is a trivial case since then $F = k/\|k\|_{A^2}$.) With these choices, Lemma 1.2 gives

$$\begin{aligned}\left|\int_{\mathbb{D}} \overline{F^{p/2}} F^{(p/2)-1}(zh(z))' d\sigma\right| &\leq C\|F^{p/2}\|_{H^{2p_1/p}} \|F^{p/2-1}\|_{H^{2p_1/(p-2)}} \|zh\|_{H^{q_1'}} \\ &= C\|F\|_{H^{p_1}}^{p/2} \|F\|_{H^{p_1}}^{(p-2)/2} \|h\|_{H^{q_1'}} \\ &= C\|F\|_{H^{p_1}}^{p-1} \|h\|_{H_{q_1'}}.\end{aligned}$$

Since

$$\left|\int_{\mathbb{D}} \overline{k(z)}(zh(z))' d\sigma\right| \leq C\|\phi\|\|F\|_{H^{p_1}}^{p-1} \|h\|_{H_{q_1'}}$$

for all polynomials h, we may define a continuous linear functional ψ on $H^{q_1'}$ such that

$$\psi(h) = \int_{\mathbb{D}} \overline{k(z)}(zh(z))' d\sigma$$

for all analytic polynomials h. Then ψ has an associated kernel in H^{q_1}, which we will call $\widetilde{k}$. Thus, for all $h \in H^{q_1'}$, we have

$$\psi(h) = \frac{1}{2\pi}\int_0^{2\pi} \overline{\widetilde{k}(e^{i\theta})} h(e^{i\theta}) d\theta.$$

But then the Cauchy-Green theorem gives

(4.2)

$$\begin{aligned}\int_{\mathbb{D}} \overline{k(z)}(zh(z))'\, d\sigma &= \psi(h) \\ &= \frac{1}{2\pi}\int_{\partial\mathbb{D}} \overline{\widetilde{k}(e^{i\theta})} h(e^{i\theta})\, d\theta = \frac{i}{2\pi}\int_{\partial\mathbb{D}} \overline{\widetilde{k}(z)} h(z) z\, d\overline{z} \\ &= \lim_{r\to 1} \frac{i}{2\pi}\int_{\partial(r\mathbb{D})} \overline{\widetilde{k}(z)} h(z) z\, d\overline{z} = \lim_{r\to 1}\int_{r\mathbb{D}} \overline{\widetilde{k}(z)}(zh(z))'\, d\sigma \\ &= \int_{\mathbb{D}} \overline{\widetilde{k}(z)}(zh(z))'\, d\sigma,\end{aligned}$$

where h is any analytic polynomial.

Now, for any polynomial $h(z)$, define the polynomial $H(z)$ so that

$$H(z) = \frac{1}{z}\int_0^z h(\zeta)\, d\zeta.$$

Then substituting $H(z)$ for $h(z)$ in equation (4.2), and using the fact that $(zH)' = h$, we have

$$\int_{\mathbb{D}} \overline{\widetilde{k}(z)} h(z) d\sigma = \int_{\mathbb{D}} \overline{k(z)} h(z) d\sigma$$

for every polynomial h. But since the polynomials are dense in A^p, and k and $\widetilde{k}$ are both in A^q, which is isomorphic to the dual space of A^p, we must have that $k = \widetilde{k}$, and thus $k \in H^{q_1}$.

Now for any polynomial h,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(e^{i\theta}) d\theta \le C\|\phi\| \|F\|_{H^{p_1}}^{p-1} \|h\|_{H^{q_1'}},$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(e^{i\theta}) d\theta \le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|h\|_{H^{q_1'}}$$

by inequality (1.1). But if h is any trigonometric polynomial,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} h(\theta) d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \overline{k(e^{i\theta})} \left[S(h)(e^{i\theta})\right] d\theta \\ &\le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|S(h)\|_{H^{q_1'}} \\ &\le C\|k\|_{A^q} \|F\|_{H^{p_1}}^{p-1} \|h\|_{L^{q_1'}}, \end{aligned}$$

where S denotes the Szegő projection. Taking the supremum over all trigonometric polynomials h with $\|h\|_{L^{q_1'}} \le 1$ and dividing both sides of the inequality by $\|k\|_{A^q}$ we arrive at the required bound. □

The main results of this section can be summarized in the following theorem.

Theorem 4.4. *Suppose that p is an even integer with conjugate exponent q. Let $k \in A^q$ and let F be the A^p extremal function associated with k. Let p_1, q_1 be a pair of numbers such that $q \le q_1 < \infty$ and*

$$p_1 = (p-1)q_1.$$

Then $F \in H^{p_1}$ if and only if $k \in H^{q_1}$. More precisely,

$$C_1 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)} \le \|F\|_{H^{p_1}} \le C_2 \left(\frac{\|k\|_{H^{q_1}}}{\|k\|_{A^q}}\right)^{1/(p-1)}$$

where C_1 and C_2 are constants that depend only on p and p_1.

Note that if $p_1 = (p-1)q_1$, then $q \le q_1 < \infty$ is equivalent to $p \le p_1 < \infty$.

5. Proof of the Lemmas

We now give the proofs of Lemmas 1.1 and 1.2. These proofs are rather technical and require applications of maximal functions and Littlewood-Paley theory.

Definition 5.1. For a function f analytic in the unit disc, the Hardy-Littlewood maximal function is defined on the unit circle by

$$f^*(e^{i\theta}) = \sup_{0\le r<1} |f(re^{i\theta})|.$$

The following is the simplest form of the Hardy-Littlewood maximal theorem (see for instance [2], p. 12).

Theorem B. (Hardy-Littlewood.) *If $f \in H^p$ for $0 < p \leq \infty$, then $f^* \in L^p$ and*

$$\|f^*\|_{L^p} \leq C\|f\|_{H^p},$$

where C is a constant depending only on p.

Further results of a similar type may be found in [5].

Definition 5.2. For a function f analytic in the unit disc, the Littlewood-Paley function is

$$g(\theta, f) = \left\{ \int_0^1 (1-r)|f'(re^{i\theta})|^2 dr \right\}^{1/2}.$$

A key result of Littlewood-Paley theory is that the Littlewood-Paley function, like the Hardy-Littlewood maximal function, belongs to L^p if and only if $f \in H^p$. Formally, the result may be stated as follows (see [11], Volume 2, Chapter 14, Theorems 3.5 and 3.19).

Theorem C. (Littlewood-Paley.) *For $1 < p < \infty$, there are constants C_p and B_p depending only on p so that*

$$\|g(\cdot, f)\|_{L^p} \leq C_p\|f\|_{H^p}$$

for all functions f analytic in $\mathbb{D}$, and

$$\|f\|_{H^p} \leq B_p\|g(\cdot, f)\|_{L^p}$$

for all functions f analytic in $\mathbb{D}$ such that $f(0) = 0$.

We now apply the Littlewood-Paley theorem to obtain the following result, from which Lemmas 1.1 and 1.2 will follow.

Theorem 5.3. *Suppose $1 < p_1, p_2 \leq \infty$, and let p be defined by $1/p = 1/p_1 + 1/p_2$. Suppose furthermore that $1 < p < \infty$. If $f_1 \in H^{p_1}$ and $f_2 \in H^{p_2}$, and h is defined by*

$$h(z) = \int_0^z f_1(\zeta) f_2'(\zeta)\, d\zeta,$$

then $h \in H^p$ and $\|h\|_{H^p} \leq C\|f_1\|_{H^{p_1}}\|f_2\|_{H^{p_2}}$, where C depends only on p_1 and p_2.

Proof. By the definitions of the Littlewood-Paley function and the Hardy-Littlewood maximal function,

$$\begin{aligned} g(\theta, h) &= \left\{ \int_0^1 (1-r)|f_1(re^{i\theta})f_2'(re^{i\theta})|^2\, dr \right\}^{1/2} \\ &\le f_1^*(\theta) \left\{ \int_0^1 (1-r)|f_2'(re^{i\theta})|^2\, dr \right\}^{1/2} \\ &= f_1^*(\theta) g(\theta, f_2). \end{aligned}$$

Therefore, since $h(0) = 0$, Theorem C gives

$$\|h\|_{H^p} \le C\|g(\cdot, h)\|_{L^p} \le C\|f_1^*\, g(\cdot, f_2)\|_{L^p}.$$

Applying first Hölder's inequality and then Theorem B, we infer that

$$\|h\|_{H^p} \le C\|f_1^*\|_{L^{p_1}} \|g(\cdot, f_2)\|_{L^{p_2}} \le C\|f_1\|_{H^{p_1}} \|g(\cdot, f_2)\|_{L^{p_2}}.$$

If $p_2 < \infty$, Theorem C allows us to conclude that

$$\|h\|_{H^p} \le C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}.$$

This proves the claim under the assumption that $p_2 < \infty$.

If $p_2 = \infty$, then $p_1 < \infty$ by assumption. Integration by parts gives

$$h(z) = f_1(z)f_2(z) - f_1(0)f_2(0) - \int_0^z f_2(\zeta) f_1'(\zeta)\, d\zeta.$$

The H^p norm of the first term is bounded by $\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$, by Hölder's inequality. The second term is bounded by $C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$ for some C, since point evaluation is a bounded functional on Hardy spaces. The H^p norm of the last term is bounded by $C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$, by what we have already shown, and thus $\|h\|_{H^p} \le C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$. □

Theorem 5.3 will now be used together with the Cauchy-Green theorem to prove Lemmas 1.2 and 1.1.

Proof of Lemma 1.2. Define

$$I_r = \int_{r\mathbb{D}} \overline{f_1} f_2 f_3'\, dA \qquad \text{and} \qquad H(z) = \int_0^z f_2(\zeta) f_3'(\zeta) d\zeta.$$

Then Theorem 5.3 says that $H \in H^q$ and that $\|H\|_{H^q} \le C\|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}}$, where $\frac{1}{q} = \frac{1}{p_2} + \frac{1}{p_3}$. By the Cauchy-Green formula,

$$I_r = \frac{i}{2} \int_{\partial(r\mathbb{D})} \overline{f_1(z)} H(z)\, d\overline{z}.$$

Since $1/p_1 + 1/q = 1$, Hölder's inequality gives

$$|I_r| = \frac{1}{2} \left| \int_{\partial(r\mathbb{D})} \overline{f_1(z)} H(z)\, d\overline{z} \right| \le \pi M_{p_1}(f_1, r) M_q(H, r).$$

But since $\|H\|_{H^q} \leq C\|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}}$, this shows that

$$|I_r| \leq C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3\|_{H^{p_3}},$$

which bounds the principal value in question, assuming it exists.

To show that it exists, note that for $0 < s < r$, the Cauchy-Green formula gives

$$\begin{aligned} 2|I_r - I_s| &= \left| \int_{\partial(r\mathbb{D} - s\mathbb{D})} \overline{f_1(z)} H(z)\, d\overline{z} \right| \\ &= \left| \int_0^{2\pi} \left[r\overline{f_1(re^{i\theta})} H(re^{i\theta}) - s\overline{f_1(se^{i\theta})} H(se^{i\theta}) \right] e^{-i\theta}\, d\theta \right| \\ &\leq \left| \int_0^{2\pi} \overline{f_1(re^{i\theta})} \left(rH(re^{i\theta}) - sH(se^{i\theta}) \right) e^{-i\theta}\, d\theta \right| \\ &\quad + \left| \int_0^{2\pi} s \left(\overline{f_1(re^{i\theta})} - \overline{f_1(se^{i\theta})} \right) H(se^{i\theta})\, e^{-i\theta}\, d\theta \right|. \end{aligned}$$

We let $f_r(z) = f(rz)$. Then Hölder's inequality shows that the expression on the right of the above inequality is at most

$$M_{p_1}(f_1, r)\|rH_r - sH_s\|_{H^q} + s\|(f_1)_r - (f_1)_s\|_{H^{p_1}} M_q(H, r).$$

Since $p_1 < \infty$ and $q < \infty$, we know that $(f_1)_r \to f_1$ in H^{p_1} as $r \to 1$, and $H_r \to H$ in H^q as $r \to 1$ (see [2], p. 21). Thus the above quantity approaches 0 as $r, s \to 1$, which shows that the principal value exists.

For the last part of the lemma, what was already shown gives

$$\begin{aligned} \text{p.v.} \int_{\mathbb{D}} \overline{f_1} f_2 f_3' d\sigma - \int_{\mathbb{D}} \overline{f_1} f_2 (S_n f_3)' d\sigma &= \text{p.v.} \int_{\mathbb{D}} \overline{f_1} f_2 (f_3 - S_n f_3)' d\sigma \\ &\leq C\|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \|f_3 - S_n(f_3)\|_{H^{p_3}}. \end{aligned}$$

By assumption $p_3 > 1$. If also $p_3 < \infty$, then the right hand side approaches 0 as $n \to \infty$, which finishes the proof. □

Proof of Lemma 1.1. We know that $f^{p/2} \in H^2$ and $f^{(p/2)-1} \in H^{2p/(p-2)}$. Since h is a polynomial, we have $f^{(p/2)-1} h \in H^{2p/(p-2)}$. Also,

$$\frac{1}{2} + \frac{p-2}{2p} + \frac{1}{p} = 1.$$

Thus, Lemma 1.2 with $f_1 = f^{p/2}$, and $f_2 = f^{(p/2)-1} h$, and $f_3 = f$ gives the result.

□

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