# Gangster Operators and Invincibility of Positive Semidefinite Matrices *

Timothy Ferguson and Charles Johnson †

October 16, 2010

**Abstract**

Decrease in absolute value of a symmetrically placed pair of off diagonal entries need not preserve positive definiteness of an $n \times n$ matrix, $n \geq 3$. A gangster operator is one that replaces some such pairs by 0's. Circumstances in which gangster operators preserve positive definiteness are investigated. Certain general circumstances are given, and graphs that ensure preservation are characterized.

## 1 Introduction

In semidefinite programming some pairs of off-diagonal entries of a positive semi-definite (PSD) matrix are replaced by 0's, and it is useful to know if the resulting matrix is necessarily PSD (see [1] and [2]). If $A$ is the original $n \times n$ matrix, the resulting matrix may simply be described via the Hadamard product $(A \circ H)$ with a certain symmetric $0, 1$ matrix $H$ (with non-zero diagonal entries), namely the one with 0's in precisely those positions in which the entries of $A$ are replaced by 0's. The transformation $A \rightarrow H \circ A$ is called a *gangster operator* (as it shoots "holes" in $A$), and we simply refer to $H$ as a gangster operator. Since the application of a gangster operator to a PSD matrix need not be PSD, this concept raises a series of natural and fundamental questions about PSD matrices.

**Definition 1.1.** The *order* of a gangster operator $H$ is the number of pairs of off-diagonal 0's in $H$, and a PSD matrix $A$ is called $k$*-fold invincible* if $H \circ A$ is PSD for any gangster operator $H$ of order $k$ or less.

**Definition 1.2.** A PSD matrix $A$ is called (fully) invincible if $A$ is $k$-fold invincible for all $k$, i.e. if any gangster operator applied to $A$ produces a PSD result.

*This research was supported by NSF Grant DMS-03-53510

†Department of Mathematics, College of William and Mary, Williamsburg, VA

It is clear that an $n \times n$ matrix is fully invincible if and only if it is $[(n^2-n)/2]$-fold invincible.

The issue of ($k$-fold) invincibility is naturally about what happens to the smallest eigenvalue, $\lambda_1$, when gangster operators are applied. Because application of a gangster operator commutes with translation by a scalar multiple of $I$, another natural concept is suggested. Let $\mathcal{G}_n$ denote the set of all $n \times n$ gangster operators.

**Definition 1.3.** An $n \times n$ Hermitian matrix $A$ is called $\lambda_1$-*minimal* if

$$\lambda_1(A) \leq \min_{H \in \mathcal{G}_n} \lambda_1(H \circ A) = \Lambda_1(A).$$

In general $g(A) = \lambda_1(A) - \Lambda_1(A) \geq 0$ is called the *gangster gap* of $A$. It is then clear that $A$ is invincible if and only if $\lambda_1(A) \geq g(A)$; any $\lambda_1$-minimal PSD matrix is fully invincible, and any PSD matrix $A$ with $\lambda_1(A) = 0$ is fully invincible if and only if it is $\lambda_1$-minimal.

In the next section, we give a number of observations that illuminate the notion of invincibility, many of which will be used repeatedly in subsequent sections. In section 3, we characterize the $\lambda_1$-minimal matrices, thereby providing several sufficient conditions for invincibility. In section 4, the number of gangster operators that need to be checked for invincibility is substantially reduced and this is applied to the case of doubly nonnegative matrices. Section 5 analyzes invincibility in certain special situations and makes general observations from the results, and section 6 considers invincibility for PSD matrices with given graphs. In particular it is shown that the only graphs for which all PSD matrices are invincible are trees. Section 7 discusses invincibility of $2 \times 2$, $3 \times 3$, and $4 \times 4$ matrices.

## 2 Elementary Observations

We now discuss some elementary facts about invincible matrices that are needed elsewhere. We first mention several operations under which the classes of fully and $k$-fold invincible matrices are invariant.

**Proposition 2.1.** *Any matrix diagonally congruent to a fully($k$-fold) invincible matrix is fully($k$-fold) invincible.*

*Proof.* If $D$ is a non-singular diagonal matrix, $H$ is a gangster operator, and $A$ is a matrix, then

$$H \circ (DAD) = D(H \circ A)D.$$

The claim then follows from the above equation and from the fact that a PSD matrix remains PSD under diagonal congruence. □

The next two observations are immediate.

**Proposition 2.2.** *If a matrix $A$ is fully($k$-fold) invincible, then any permutation similarity of $A$ is fully($k$-fold) invincible.*

**Proposition 2.3.** *If* $0$ *denotes the all zero* $m \times m$ *matrix and* $A$ *is a PSD matrix, then* $A$ *is fully(k-fold) invincible if and only if* $A \oplus 0$ *is fully(k-fold) invincible.*

We now mention a helpful fact that follows from Propositions 2.1, 2.2 and 2.3: *In investigating properties of invincibility, we may assume that all the diagonal entries of the matrices we work with are positive, and in particular* 1. To see that this is true, note that any PSD matrix $A$ with a zero entry in the $i^{\text{th}}$ diagonal position must have all 0 entries in both row $i$ and column $i$; then a permutation can put the matrix into the form $A' \oplus 0$, and the original matrix $A$ will be fully($k$-fold) invincible if and only if $A'$ is. By repeating this process, we may assume that $A$ has no zero diagonal entries. *We now assume that all matrices have positive diagonal entries, unless otherwise noted.*

If we now use the appropriate diagonal congruence we see that we may assume that all the diagonal entries of $A$ are 1. Because of this, we will sometimes state results for matrices with all 1's down the diagonal only.

We next have the following proposition, which leads to several useful observations.

**Proposition 2.4.** *The set of all* $n \times n$ *fully(k-fold) invincible matrices form a closed cone.*

*Proof.* Since

$$H \circ (\alpha A + (1-\alpha)B) = \alpha(H \circ A) + (1-\alpha)(H \circ B),$$

both claims are a simple consequence of the fact that the set of PSD matrices form a closed cone. □

We now have the following two corollaries.

**Corollary 2.5.** *In any k-fold invincible matrix, any k (symmetric) pairs of entries can be simultaneously multiplied by any k independent real scalars between* 0 *and* 1 *inclusive and the resulting matrix will be positive semidefinite.*

*Proof.* Let $0 \leq \alpha \leq 1$. To show the claim for one-fold invincibility, let $A$ be any one-fold invincible matrix, and replace one pair of entries with 0, to get the matrix $A'$. Then $\alpha A + (1-\alpha)A'$ is the same matrix as $A$, except the given pair of entries is multiplied by $\alpha$. Also, both $A$ and $A'$ are PSD, so $\alpha A + (1-\alpha)A'$ is PSD as well.

Now, assume the claim is true for $(k-1)$-fold invincibility. Let $A$ be a $k$-fold invincible matrix, let $H$ be a gangster operator consisting of all $1's$ except for one pair of 0 entries. Let $A' = H \circ A$. Then $\alpha A + (1-\alpha)A'$ is the same matrix as $A$, except the given pair of entries is multiplied by $\alpha$. But now $\alpha A + (1-\alpha)A'$ is $(k-1)$-fold invincible, since $A$ is $k$-fold invincible and $A'$ is $(k-1)$-fold invincible. Thus, the claim is true by induction. □

As a special case, we have the following.

**Corollary 2.6.** *In any fully invincible matrix, all symmetrically placed pairs of entries can simultaneously be multiplied by independent scalars between* $0$ *and* $1$ *inclusive and the result will be fully invincible.*

We now mention some facts relating to Hadamard products of invincible matrices.

**Proposition 2.7.** *If* $A$ *is* $k$*-fold invincible and* $B$ *is* $m$*-fold invincible, then* $A \circ B$ *is (at least)* $(k+m)$*-fold invincible.*

*Proof.* Let $G$ be a gangster operator of order at most $k+m$. Then we can write $G = G_k \circ G_m$, where $G_k$ has order at most $k$ and $G_m$ has order at most $m$. Now we see that $G(A \circ B) = G_k(A) \circ G_m(B)$, which is positive definite by the Schur product theorem ([3] 458). □

**Corollary 2.8.** *If* $A$ *and* $B$ *are invincible, so is* $A \circ B$.

# 3 Essentially $Z$-matrices and Diagonal Dominance

Recall that the matrices $A$ and $B$ are said to be signature similar if there is a diagonal matrix $D$, all of whose entries are $\pm 1$, such that $A = DBD$.

**Definition 3.1.** A $Z$-matrix is one in which all off-diagonal entries are non-positive. An essentially $Z$ matrix is a matrix that is signature similar to a $Z$-matrix.

We now have the following theorem, which characterizes real $\lambda_1$-minimal matrices.

**Theorem 3.2.** *If* $A$ *is a real irreducible symmetric matrix, then* $A$ *is* $\lambda_1$*-minimal if and only if* $A$ *is an essentially* $Z$*-matrix.*

*Proof.* First we show that any essentially $Z$-matrix is $\lambda_1$-minimal. Let $A$ be an essentially $Z$-matrix, so that $A = DBD$, where $D$ is a diagonal matrix with all entries $\pm 1$, and $B$ a $Z$ matrix. Thus we may write $B = bI - P$, where $P$ is a nonnegative matrix, and $b$ is the largest diagonal entry of $B$. Now, if $H$ is any gangster operator, $H \circ P \leq P$, where $C \leq D$ means that $c_{ij} \leq d_{ij}$ for all $i, j$. Thus, by Corollary 8.1.19 in [3], we have that $\rho(H \circ P) \leq \rho(P)$. Now, by Theorem 8.3.1 in [3], we have that $\rho(P)$ is an eigenvalue of $P$; clearly $\rho(P)$ is the maximum eigenvalue of $P$. Similarly, $\rho(H \circ P)$, is the maximum eigenvalue of $H \circ P$. Thus, applying any gangster operator to $P$ does not increase its largest eigenvalue, and so applying any gangster operator to $B$ will not decrease its smallest eigenvalue. So if $H$ is any gangster operator, then using the fact that eigenvalues are invariant under signature similarity we have that

$$\lambda_1(A) = \lambda_1(B) \leq \lambda_1(H \circ B) = \lambda_1(D(H \circ B)D) = \lambda_1(H \circ [DBD]) = \lambda_1(H \circ A).$$

To show that any irreducible matrix that is $\lambda_1$-minimal is essentially a $Z$-matrix, let the matrix $A$ be irreducible, $\lambda_1$-minimal, and not essentially a $Z$-matrix. Now, if $\beta$ is any scalar, $A$ is $\lambda_1$-minimal if and only if $A+\beta I$ is $\lambda_1$-minimal. With the appropriate value of $\beta$, the matrix $A+\beta I$ will have minimum eigenvalue 0, so we may assume without loss of generality that $A$ has minimum eigenvalue 0. Suppose $x$ is an eigenvector corresponding to 0. By permutations and signature similarity, we may assume without loss of generality that either $x$ is entrywise positive or that $x = \left(\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right)$, where $w > 0$.

First suppose that $x = \left(\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right)$. Write $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{smallmatrix}\right)$, where $A_{11}$ is a square matrix with the same number of rows as $w$. The eigenvalue equation shows that $A_{11}w = A_{12}^T w = 0$. Also, $A_{12} \neq 0$ since $A$ is irreducible. Thus, there must be at least one positive element of $A_{12}$, since $A_{12}w^T = 0$, and $w$ is entry-wise greater than 0. Replace at least one of the positive elements in $A_{12}$ by 0 to form a matrix $\tilde{A}$. Then, letting $e$ represent the vector with all entries 1, we see that

$$\begin{aligned} \alpha_{\tilde{A}}(\epsilon) &= \begin{pmatrix} w^T & \epsilon e^T \end{pmatrix} \begin{pmatrix} A_{11} & \tilde{A_{12}} \\ \tilde{A_{12}}^T & A_{22} \end{pmatrix} \begin{pmatrix} w \\ \epsilon e \end{pmatrix} \\ &= w^T A_{11} w + 2\epsilon w^T \tilde{A_{12}} e + \epsilon^2 e^T A_{22} e \\ &= 2\epsilon w^T \tilde{A_{12}} e + \epsilon^2 e^T A_{22} e. \end{aligned}$$

Since $w^T A_{12} e = 0$, we know that $w^T \tilde{A}_{12} e < 0$. Because of this, the linear term of $\alpha_{\tilde{A}}$ is negative so $\alpha_{\tilde{A}}(\epsilon)$ is negative for all sufficiently small $\epsilon > 0$. Thus $\tilde{A}$ has a negative eigenvalue, contradicting our assumption.

Thus, we may assume without loss of generality that $A$ has an entry-wise positive eigenvector $x$ corresponding to the minimum eigenvalue of $A$. But since $A$ has off diagonal entries that are positive, letting $H$ be a gangster operator that replaces these entries by 0, we will have that $x^T(H \circ A)x < x^T Ax$, so that the smallest eigenvalue of $H \circ A$ is less than the smallest eigenvalue of $A$, contrary to our assumption. □

Recall that a positive definite $Z$-matrix is an $M$-matrix. When we say that a symmetric matrix is "essentially an $M$-matrix," we mean that it is signature similar to an $M$-matrix.

**Corollary 3.3.** *Any symmetric matrix that is essentially an $M$-matrix is fully invincible.*

This follows from the theorem since invincibility is invariant under diagonal congruence and thus signature similarity.

**Corollary 3.4.** *If $A$ is a real, symmetric, PSD matrix and* $\lambda_1(A) = 0$, *(that is, $A$ is positive semidefinite but not positive definite), then $A$ is fully invincible if and only if $A$ is signature similar to a $Z$-matrix.*

It is well known that $M$-matrices are diagonally equivalent to matrices that are diagonally dominant and symmetric $M$-matrices are diagonally congruent to diagonally dominant matrices. Full invincibility occurs not just for $M$-matrices, but also for any diagonally dominant matrix.

**Theorem 3.5.** *Let $A$ be a symmetric matrix that has all positive diagonal entries. Then, if $A$ is row (or column) weakly diagonally dominant, $A$ is fully invincible.*

*Proof.* All of the eigenvalues of $A$ must lie in the closed right half of the complex plane, by Gerŝgorin's Theorem ([3] 6.1.1) and so the matrix must be positive semidefinite. Now, applying a gangster operator to a weakly diagonally dominant matrix leaves it weakly diagonally dominant, so the result follows. $\square$

**Corollary 3.6.** *If a symmetric matrix with positive diagonal entries is diagonally congruent to a diagonally dominant matrix, then it is fully invincible.*

Recall that the *comparison matrix* of a given matrix is the matrix obtained from the original one by replacing all of its diagonal entries by their absolute values, and all of its off diagonal entries by the negative of their absolute values. We now have the following theorem.

**Theorem 3.7.** *Any symmetric matrix with positive diagonal entries whose comparison matrix is an $M$-matrix is fully invincible.*

*Proof.* For completeness, we first show that $M$-matrices are diagonally congruent to weakly diagonally dominant matrices. Let $A$ be an $M$-matrix. Then from [4] 2.5.3.12, $Ax = y > 0$ for some component wise positive vectors $x$ and $y$. This can be rewritten as $AD_xe = y$, where $D_x$ is the diagonal matrix whose diagonal entries are the entries of $x$, and $e$ is the vector consisting of all 1's. But this implies that $AD_x$ will have all its row sums greater than zero. Then $D_xAD_x$ will have all of its row sums greater than zero as well. But since $D_xAD_x$ will have the $Z$ sign pattern, this implies that $D_xAD_x$ is row diagonally dominant.

Now, if $A$ is not necessarily an $M$ matrix but has comparison matrix $M(A)$ that is an $M$-matrix, then $DM(A)D$ will diagonally dominant for some diagonal matrix $D$, where $D$ can be chosen to be positive. But $M(DAD) = DM(A)D$, so $M(DAD)$ and thus $DAD$ will be diagonally dominant, so $A$ will be fully invincible. $\square$

However, not every invincible matrix is diagonally dominant. For example, the matrix

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

is invincible, which we can see by checking directly or applying theorem 7.1. However, it is not diagonally dominant.

## 4 Eliminating Certain Gangster Operators when Checking for Invincibility

*A priori*, the definition of invincible matrices requires that, when checking if a PSD matrix is invincible, we check whether it remains PSD after applying each possible gangster operator to it. However, in this section we show that it is only necessary to check a certain subset of the gangster operators.

**Definition 4.1.** Let $S_n$ denote the set of real symmetric $n \times n$ matrices.

- Let $K \subset \{1, 2, \ldots, n\}$. Let $A = (a_{ij}) \in S_n$. A *plus-minus gangster operator for $A$ with index set $K$* is a gangster operator $H = (h_{ij})$ such that

  1. If $i \neq j$, and if both $i$ and $j$ are elements of $K$ or both are elements of $K^c$, and if $a_{ij} > 0$, then $h_{ij} = 0$.
  2. If $i \neq j$, and if one of $i$ and $j$ belongs to $K$ and the other belongs to $K^c$, and $a_{ij} < 0$, then $h_{ij} = 0$.
  3. For all other cases, $h_{ij} = 1$.

  When the matrix $A$ is understood we sometimes simply call $H$ a plus-minus gangster operator.

- Let $A \in S_n$ such that all entries of $A$ are positive. If $H$ is a plus-minus gangster operator for $A$, we say that $H$ is a bipartite gangster operator.

Note that the plus-minus gangster operator (for some $n \times n$ matrix $A$) with index set $K$ is the same as the plus-minus gangster operator for that matrix with index set $K^c$. Note also that the sign pattern of $A \in S_n$ determines what all the plus-minus gangster operators for $A$ are. Since the sign pattern is given in the definition of bipartite gangster operators, which matrix $A$ we choose in their definition is irrelevant.

The reason for this definition is the following theorem.

**Theorem 4.2.** *For a given real symmetric matrix $A$, there is a plus-minus gangster operator $H$ for $A$ such that*

$$\min_{G \in \mathcal{G}} \lambda_1(G \circ A) = \lambda_1(H \circ A).$$

*Ie., if $\mathcal{H}$ is the set of all plus-minus gangster operators, then*

$$\min_{G \in \mathcal{G}} \lambda_1(G \circ A) = \min_{H \in \mathcal{H}} \lambda_1(H \circ A).$$

*Proof.* Let $S_n$ denote the set of real symmetric $n \times n$ matrices. Recall that $\lambda_1(B) = \min_{\substack{x \in \mathbb{R}^n \\ x^T x = 1}} x^T B x$ for any $B \in S_n$.

Now, let $A \in S_n$. Let $z \in \mathbb{R}^n$ be such that, for some gangster operator $H$,

$$z^T (H \circ A) z = \min_{\substack{G \in \mathcal{G}_n, w \in \mathbb{R}^n \\ w^T w = 1}} w^T (G \circ A) w. \tag{4.1}$$

Then we will have that

$$z^T(H \circ A)z = \min_{G \in \mathcal{G}_n} \lambda_1(G \circ A).$$

Now, permute $z$ to $z'$ so that

$$z' = \left(\begin{smallmatrix} x \\ -y \end{smallmatrix}\right),$$

where both $x$ and $y$ are component-wise nonnegative, and permute $A$ to $A'$ by applying the same permutation used on $z$ to both rows and columns of $A$. Then

$$\begin{pmatrix} x^T & -y^T \end{pmatrix} \begin{pmatrix} A'_{11} & A'_{12} \\ A'^T_{12} & A'_{22} \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = x^T A'_{11} x + y^T A'_{22} y - 2x^T A'_{12} y.$$

Now, the gangster operator $G^*$ that replaces all positive entries of $A'_{11}$ and $A'_{22}$ with 0 and all negative entries of $A'_{12}$ with 0 will minimize the quadratic form $z'^T(G \circ A')z'$, where the minimum is taken over all gangster operators. Now, applying the reverse permutation (from the one that took $A$ to $A'$) to $G^*$ will yield a plus-minus gangster operator $H'$ for $A$ with the property that

$$\min_{G \in \mathcal{G}} z^T(G \circ A)z = z^T(H' \circ A)z.$$

But now from this equation and equation (4.1), we may identify $H'$ with $H$ from equation (4.1), and we see that

$$z^T(H' \circ A)z = \min_{G \in \mathcal{G}_n} \lambda_1(G \circ A).$$

The theorem then follows since $H'$ is a plus-minus gangster operator for $A$. $\square$

Note that one should not neglect to check plus-minus gangster operators whose index set is empty (or equivalently, $\{1, 2, \ldots, n\}$).

We also have the following corollary, which is more limited in scope than the theorem, but easier to apply. First, recall that a *doubly nonnegative matrix* is a positive semidefinite matrix, all of whose entries are nonnegative.

**Corollary 4.3.** *A doubly nonnegative matrix is fully invincible if and only if $H \circ A$ is PSD for each bipartite gangster operator $H$.*

# 5 Invincibility of Matrices with Special Form and Implications

We now consider the question of invincibility in special cases. As we shall see, investigating these special cases leads to some general results.

We first consider real matrices of the form

$$A(t) = \begin{pmatrix} 1 & t & \ldots & t \\ t & 1 & \ldots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \ldots & 1 \end{pmatrix}, \ 0 \le t \le 1.$$

Note that $A(t) = (1-t)I + tJ$. We require $t \leq 1$ since otherwise the matrix $A(t)$ would not be PSD, and we require $t \geq 0$ since if $t < 0$ the matrix $A(t)$ is a $Z$-matrix and thus is invincible if and only if it is PSD (see theorem 3.2).

**Proposition 5.1.** *For each $n$, there is a constant $T_n > 0$ such that (for $t \geq 0$) the $n \times n$ matrix $A(t)$ is fully invincible if and only if $t \leq T_n$.*

*Proof.* First, note that $A(0)$ is invincible, but $A(t)$ is not invincible for $t > 1$, (since in this case it is not PSD). Observe that if a $A(t)$ is fully invincible, then $A(\tilde{t})$ will also be, if $\tilde{t} \leq t$. This can be seen by noting that if $A(t)$ is fully invincible, so is

$$A(\tilde{t}) = (\tilde{t}/t)A(t) + (1 - (\tilde{t}/t))I,$$

using the fact that the invincible matrices form a cone. This shows that there is a $T_n$ between 0 and 1 such that $A(t)$ is invincible if $t < T_n$ and not invincible if $t > T_n$. But since the cone of invincible matrices is closed, we have that $A(T_n)$ is invincible as well.

Now, all we must do is show that $T_n > 0$ in order to prove the theorem. But $A(1/n)$ is diagonally dominant and so must be invincible by Theorem 3.5. This shows that $T_n \geq 1/n$, so we are done. □

Now that we have show the existence of $T_n$, we wish to find its value. Fortunately, this is possible.

**Theorem 5.2.** $T_n = (\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil)^{-1/2}$.

*Proof.* Let $J_{n,m}$ denote the $n \times m$ matrix with each entry 1, and let $J_m = J_{m,m}$. For fixed $n$, and $m < n$, define

$$B_m = \begin{pmatrix} 0 & J_{m,n-m} \\ J_{n-m,m} & 0 \end{pmatrix}.$$

We see that

$$B_m^2 = \begin{pmatrix} J_{m,n-m}J_{n-m,m} & 0 \\ 0 & J_{n-m,m}J_{m,n-m} \end{pmatrix} = \begin{pmatrix} (n-m)J_m & 0 \\ 0 & mJ_{n-m} \end{pmatrix}.$$

Now if $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ is any eigenvector of $B_m^2$, then we have

$$B_m^2 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (n-m)J_m X \\ mJ_{n-m}Y \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix},$$

so $X$ is an eigenvector of $J_m$ and $Y$ is an eigenvector of $J_{n-m}$. But the eigenvalues of $J_m$ are 0 and $m$, so we have that either

$$\lambda X = (n-m)J_m X = m(n-m)X$$

or

$$\lambda X = (n-m)J_m X = 0.$$

This implies that $\lambda = m(n-m)$ or 0. So the largest possible eigenvalue of a $B_m^2$ matrix will be at most $\lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor)$. Thus, the smallest possible eigenvalue for a $B_m$ matrix will be

$$-\sqrt{\lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor)} = -\sqrt{\lfloor n/2 \rfloor \lceil n/2 \rceil}.$$

For $n$ even, the matrix

$$\begin{pmatrix} 0 & J_{n/2} \\ J_{n/2} & 0 \end{pmatrix} \text{ with eigenvector } \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix},$$

where the eigenvector has $\frac{n}{2}$ 1's and $\frac{n}{2}$ $-1$'s, has $-\frac{n}{2}$ for an eigenvalue, so we know the minimum possible eigenvalue for the $B_m$ matrices is achieved for even $n$. For $n = 2l+1$ odd, the minimum possible eigenvalue is $-\sqrt{l(l+1)}$ and the matrix

$$\begin{pmatrix} 0 & J_{l+1,l} \\ J_{l,l+1} & 0 \end{pmatrix} \text{ has eigenvector } \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -\frac{\sqrt{l(l+1)}}{l} \\ \vdots \\ -\frac{\sqrt{l(l+1)}}{l} \end{pmatrix},$$

with eigenvalue $-\sqrt{l(l+1)}$, where the eigenvector has $l+1$ "1" entries and $l$ "$-\sqrt{l(l+1)}/l$" entries. Thus, the minimum of $-\sqrt{l(l+1)}$ is actually achieved in the odd case. Thus, in both cases we see that the minimum eigenvalue for all $n \times n$ matrices of the form $B_m$ is $\sqrt{\lfloor n/2 \rfloor \lceil n/2 \rceil}$.

Now, by symmetry and Corollary 4.3, the matrix $A(t)$ will be invincible if and only if every matrix of the form

$$\begin{pmatrix} I_m & tJ_{m,n-m} \\ tJ_{n-m,m} & I_{n-m} \end{pmatrix}$$

is PSD. But

$$\begin{pmatrix} I_m & tJ_{m,n-m} \\ tJ_{n-m,m} & I_{n-m} \end{pmatrix} = I + tB_m.$$

Now, the smallest eigenvalue that any $B_m$ can have is exactly $-\sqrt{\lfloor n/2 \rfloor \lceil n/2 \rceil}$, so the matrix $I + tB_m$ will be PSD for all $m$ if and only if

$$t \leq \left( \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil \right)^{-1/2}.$$

□

The next result sets bounds on $t$ that require $A(t)$ to be $k$-fold invincible. However, unlike the bounds derived in the fully invincible case, these bounds are not necessarily sharp.

**Theorem 5.3.** *If*

$$A(t) = \begin{pmatrix} 1 & t & \dots & t \\ t & 1 & \dots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \dots & 1 \end{pmatrix}$$

*where* $t \leq \frac{1}{\sqrt{2k}+1}$, *then* $A(t)$ *is* $k$*-fold invincible.*

*Proof.* $A(t) = (1-t)I + tJ$ has eigenvalues $\underbrace{1-t, 1-t, \dots, 1-t}_{n-1 \text{ times}}, 1 + (n-1)t$.

Now, let $E$ be a symmetric matrix having all entries 0 except for at most $2k$ entries of value $-t$. Then $||E||_2 \leq (\sqrt{2k})t$, so by Corollary 6.3.8 in [3], we must have that $|\hat{\lambda}_1 - \lambda_1| = |\hat{\lambda}_1 - (1-t)| \leq (\sqrt{2k})t$ where $\hat{\lambda}_1$ is the smallest eigenvalue of $A(t) + E$, and $\lambda_1$ is the smallest eigenvalue of $A(t)$. So if

$$(\sqrt{2k})t \leq 1 - t \tag{5.1}$$

then $\hat{\lambda}_1$ will be nonnegative. Now, this will be true if and only if

$$t \leq \frac{1}{\sqrt{2k}+1}. \tag{5.2}$$

□

Using Theorems 5.2 and 5.3, we can now give a condition under which $k$-fold invincibility will *not* suffice for full invincibility.

**Theorem 5.4.** *For* $n \times n$ *matrices,* $k$*-fold invincibility will not suffice for full invincibility if* $\frac{1}{\sqrt{2k}+1} > T_n = (\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil)^{-1/2}$.

*Proof.* In this case there will be a matrix $A(t)$ that is $k$-fold invincible but not fully invincible. □

We now state some implications of Theorem 5.2.

**Corollary 5.5.** *If* $A = (a_{ij})$ *is an invincible* $n \times n$ *matrix, possibly with complex entries, for which* $a_{ii} = 1$ *for all* $i$*, we have that*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij} \leq (n^2 - n)T_n. \tag{5.3}$$

*Proof.* Assume that $A$ is $n \times n$ and invincible. Consider the matrix

$$B = \frac{1}{|\Pi(n)|} \sum_{\sigma \in \Pi(n)} \sigma(A),$$

where $\Pi(n)$ is the set of all permutations on $\mathbb{Z}_n$, and $\sigma(A)$ denotes applying the permutation $\sigma$ to both the rows and columns of $A$. Then $B$ is invincible since the invincible matrices form a cone. But from the definition of $B$, we see that applying any permutation in $\Pi(n)$ to $B$ leaves it unchanged, so all off diagonal entries of $B$ must be equal. Also, permuting the off diagonal elements of $A$ do not change their average; from this we see that $B$ must have the same average of off diagonal entries as $A$ does, so all of its off diagonal entries must equal the average off diagonal entry of $A$, which is real since $A$ is Hermitian. Applying Theorem 5.2 now gives the result. □

**Corollary 5.6.** *If $A = (a_{ij})$ is an $n \times n$ invincible matrix (not necessarily real), and $A$ has all of its diagonal entries equal to 1, then*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} (|a_{ij}|^2) \leq (n^2 - n)T_n. \tag{5.4}$$

*Proof.* If $A$ is invincible, so is $\overline{A}$, and thus so is $B = A \circ \overline{A} = (|a_{ij}|^2)$, by Corollary 2.8. Now, the previous corollary applies to $B$. □

**Corollary 5.7.** *Let $A$ be an $n \times n$ real symmetric matrix with all diagonal entries equal to 1. If the upper right $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ submatrix of some permutation similarity of $A$ has average entry more than $T_n$, the matrix $A$ is not fully invincible.*

*Proof.* Suppose that $A$ were fully invincible. Then the permutation similarity in question is also fully invincible. Now, form a matrix $B$ by applying to this permuted matrix the gangster operator

$$\begin{pmatrix} I_k & J_{k,l} \\ J_{l,k} & I_l \end{pmatrix}.$$

where where $k = \lfloor \frac{n}{2} \rfloor$ and $l = \lceil \frac{n}{2} \rceil$. Taking all the permutations of $B$ that leave both the $k \times l$ block in the upper right hand corner and the $l \times k$ block in the lower left hand corner in place, and averaging them will give a matrix of the form

$$\begin{pmatrix} I_k & aJ_{k,l} \\ aJ_{l,k} & I_l \end{pmatrix},$$

where $a$ is the average of all the entries in the principal submatrix of $A$ in question. Now, by the proof of Theorem 5.2, this matrix will not be positive semidefinite if $a > T_n$. □

**Corollary 5.8.** *Let $n \geq 4$. If $A$ is an $n \times n$ real symmetric matrix that is equal to $A(T_n)$, except that one pair of entries is bigger than $T_n$ and another pair of entries is different from $T_n$, then the matrix is not fully invincible.*

*Proof.* Permute the matrix so that the second pair of entries is in the $1, 2$ position and the first pair of entries is in one of the off diagonal blocks mentioned in the above theorem. $\square$

Note that this corollary gives new information only in the case when the second pair of entries is smaller than $T_n$.

**Theorem 5.9.** *Let $A$ be a complex fully invincible matrix with $1$'s on the diagonal. Then the sum of the squares of the absolute values of the off diagonal entries of any row or column of $A$ is less than or equal to $1$.*

*Proof.* Without loss of generality, we may show this for the first row and column only. We apply a gangster operator to $A$ so that we obtain the matrix $\tilde{A}$, where

$$\tilde{A} = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ \overline{a_{12}} & 1 & \mathbf{0} & 0 \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \overline{a_{1n}} & 0 & \mathbf{0} & 1 \end{pmatrix}$$

Now, a straightforward calculation shows that

$$\det \tilde{A} = 1 - \sum_{i=2}^{n} |a_{1i}|^2,$$

and this determinant must be greater than or equal to 0 for $A$ to be fully invincible. $\square$

**Corollary 5.10.** *Let $A$ be as in the previous theorem. Then the sum of the absolute value of the off diagonal entries in a given row or column of $A$ is less than or equal to $\sqrt{n-1}$.*

*Proof.* This follows from the fact that a sequence of $n$ complex numbers with $\ell^2$ norm less than or equal to 1 has $\ell^1$ norm at most $\sqrt{n}$. $\square$

# 6 Invincibility of Matrices with a Given Graph

Recall that the graph of an $n \times n$ symmetric matrix $A = (a_{ij})$ is a graph with $n$ vertices, with an edge between vertex $i$ and vertex $j$ if and only if $i \neq j$ and $a_{ij} \neq 0$. An interesting question is whether the graph of a PSD matrix tells us anything about whether it is invincible. By applying an appropriate gangster operator to the matrix $A(T_n)$, we can find an invincible matrix with any graph we choose, so we cannot tell if a matrix is not invincible by looking at its graph. However, as the next theorem shows, if a PSD matrix has a tree for a graph, then it is invincible. Also, for any graph that is not a tree, some PSD matrix will have that graph and not be invincible, so that trees are the only graphs that guarantee a PSD matrix to be invincible.

**Theorem 6.1.** *Let $G$ be a connected graph. If $G$ is a tree, all real positive semi-definite matrices with graph $G$ are fully invincible, and in fact are $\lambda_1$-minimal. If $G$ is not a tree, then there is a real PSD matrix with graph $G$ that is not 1-fold invincible.*

*Proof.* We first show that if the graph of a positive semidefinite matrix is a tree, then it is fully invincible and $\lambda_1$-minimal. To do this, we will show that any matrix with nonnegative diagonal entries whose graph is a tree is signature similar to its own comparison matrix. Thus, any positive semidefinite matrix whose graph is a tree will be fully invincible and $\lambda_1$-minimal by Theorem 3.2.

We may attach a positive or negative sign to each edge in the graph of a tree depending on whether or not the entry corresponding to that edge is positive or negative. Then applying a signature similarity using a diagonal matrix whose only negative entry is at $(i, i)$ will have the effect of switching the signs of all the edges containing vertex $i$ in the graph of the matrix, and switching only those signs.

We claim, that given any graph $G$ that is a tree, any matrix with nonnegative diagonal entries having this graph is signature similar to its comparison matrix. We prove this claim by induction on the number of vertices in the tree. The claim is clearly true for a tree with 1 vertex (this corresponds to the case of $1 \times 1$ matrices). Now, suppose the claim is true for trees with $k$ vertices. Consider an arbitrary tree with $k+1$ vertices. Now, any tree must have at least one vertex $v$, such that only one edge contains that vertex. (If not, we could easily construct a cycle in the graph of the tree). Without loss of generality we may assume this is vertex $k+1$, and that the edge containing it also contains vertex $k$. Then if we remove vertex $k+1$ and the edge containing it, we get a tree of $k$ vertices for which the claim is true. This implies that if we have a $(k+1) \times (k+1)$ matrix $A$ whose graph is a tree, then its upper left $k \times k$ sub-matrix can be changed by a signature similarity into the comparison matrix of the upper left sub-matrix. When we apply the same signature similarity to $A$, we thus get a matrix whose upper $k \times k$ sub-matrix is its own comparison matrix. Call this matrix $A'$. If entry $(k, k+1)$ of $A'$ is negative, then $A'$ is a $Z$-matrix. If that entry is positive, a signature similarity with the diagonal matrix whose only negative entry is at $(k+1, k+1)$ makes it into a $Z$-matrix. Thus, either way, $A$ is signature similar to its own comparison matrix. This proves the claim, and the fact that any PSD matrix whose graph is a tree is fully invincible and $\lambda_1$-minimal.

Now, we show that for each $n$, there exists a positive definite but not 1-fold invincible matrix whose graph is an $n$-cycle. Let $a \geq 0$, and consider the matrix $A_a$ with values $-a$ on the super-diagonal and the sub-diagonal, and 0 entries elsewhere. Now, using the Perron-Frobenius Theorem ([3] 8.4.4) and the fact that $-A_a$ is irreducible, we know that the largest eigenvalue of $-A_a$, and thus the smallest eigenvalue of $A_a$, is both algebraically and geometrically simple, and the eigenvector corresponding to this eigenvalue is component-wise positive. This means that for some matrix $\tilde{A}_a$, where $\tilde{A}_a$ is identical to $A_a$ except that its lower left and upper right entries are $\epsilon$, where $\epsilon > 0$, we have that $\tilde{A}_a$ has eigenvector $\tilde{x}$ corresponding to its smallest eigenvalue, where $\tilde{x}_i > 0$

for all $i$. Then it is not hard to see that

$$\tilde{x}^T A_a \tilde{x} < \tilde{x}^T \tilde{A}_a \tilde{x}$$

so that the smallest eigenvalue of $A_a$ is strictly less than the smallest eigenvalue of $\tilde{A}_a$.

Now, consider

$$f(a) = \max_{0 \le \epsilon \le 1} \lambda_1(\tilde{A}_a + I)$$

as a function of $a$, where $a$ is as above in the definition of $A_a$ and $\tilde{A}_a$. By the above statements, this function is strictly greater than $\lambda_1(A_a + I)$. Now, $\lambda_1(A_0+I) > 0$, but $\lambda_1(A_1+I) \le -\sqrt{2}+1 < 0$, which can be seen by considering the $3 \times 3$ case and using the interlacing inequalities. Thus, for some $a$, we will have that $f(a) > 0$ and $\lambda_1(A_a+I) < 0$, since otherwise both $f(a)$ and $\lambda_1(A_a+I)$ would both have to equal 0 for some $a$, which is impossible. Thus, there is some positive definite matrix $\tilde{A}_a$ whose graph is a cycle, but which is not positive semidefinite after applying the gangster operator that replaces the $(1, n)$ and $(n, 1)$ entries by 0.

Lastly, we must show that for any graph that contains an induced cycle, there is some positive definite matrix having that graph that is not 1-fold invincible. (Note that only trees do not contained induced cycles.) To show such a matrix exists, suppose we are given a graph $G$ with an induced cycle. Construct a matrix $A$ with 1's on the diagonal whose entries that correspond to the edges of the induced cycle are equal to the entries of some matrix with 1's on the diagonal whose graph is that cycle, and that is positive definite but not 1-fold invincible, which can be done as we have just shown. Now, set the entries of $A$ corresponding to edges of the graph that are not in the cycle equal to $\epsilon'$, for some $\epsilon' > 0$. Now, if $\epsilon'$ were 0, we would have a matrix that was positive definite but not one-fold invincible, so for small enough $\epsilon'$, we will also have a matrix that is positive definite and not 1-fold invincible, since the effect of the $\epsilon'$ entries on the eigenvalues can be made arbitrarily small. Now, this matrix has the given graph as its graph. □

The theorem and the ideas behind it have several consequences of note. A connected graph that is not a tree must contain cycles. Once these cycles are broken, the remaining graph is at most a tree, and if the resulting matrix is PSD at that point, then the resulting matrix is fully invincible. We also have the following.

**Corollary 6.2.** *If $G$ is a connected graph on $n$ vertices and $n + q - 1$ edges, then a (possibly complex) PSD matrix $A$, with $G(A) = G$, is fully invincible if and only if it is $q$-fold invincible.*

*Proof.* Suppose we apply a gangster operator $H$ of order more than $q$ to $A$, to get the matrix $A'$. Then $G(A')$ has $n$ vertices but less than $n - 1$ edges, and so is disconnected. Then $A'$ is a direct sum of the principal submatrices corresponding to the connected components of $G(A')$. Let $B'$ be one such principal submatrix of $A'$, and let $B$ be the corresponding submatrix of $A$. Then $G(B')$ is connected, and so $H$ can affect at most $q$ entries of $B$, say $a_{i_1j_1}, \ldots a_{i_mj_m}$ (where we consider the entries as belonging to $A$.) But then there is some gangster operator $H'$ of order at most $q$ that when applied to $A$, affects precisely $a_{i_1j_1}, \ldots a_{i_mj_m}$. But $H' \circ A$ is PSD by assumption, so $B' = H \circ B$ is PSD since a principal submatrix of a PSD matrix is PSD. Thus, we know that $H \circ A$ is a direct sum of PSD matrices, and so is PSD. □

Note that the proof of this corollary provides a new proof of the fact that any PSD matrix whose graph is a tree is fully invincible.

If, for example, $q = 1$, we may narrow the gangster operators that are needed to test for full invincibility. They are simply all those gangster operators that break the single cycle in $G$, which is a tree plus one edge.

**Corollary 6.3.** *If $G$ is a connected graph on $n$ vertices, with $n$ edges, and if the cycle in $G$ has $p$ edges, then there are $p$ first order gangster operators $H$ such that any PSD matrix, with $G(A) = G$, is fully invincible if and only if each $H \circ A$ is PSD.*

# 7 Invincibility for $n = 2, 3, 4$.

In this section, we investigate the problem of determining invincibility for PSD matrices of dimensions 2, 3, and 4.

Any $2 \times 2$ PSD matrix is fully invincible. There are two main ways of approaching the problem for $3 \times 3$ matrices. The first approach works for complex matrices, while the second works only for real matrices, but generalizes well to the $4 \times 4$ case.

Consider the $3 \times 3$ Hermitian matrix

$$A = \begin{pmatrix} 1 & a & b \\ \overline{a} & 1 & c \\ \overline{b} & \overline{c} & 1 \end{pmatrix}. \tag{7.1}$$

Suppose that $A$ is positive semidefinite. Consider the first order gangster operator that sets the $(1, 3)$ and $(3, 1)$ entries of the the matrix to zero, leaving

$$\tilde{A} = \begin{pmatrix} 1 & a & 0 \\ \overline{a} & 1 & c \\ 0 & \overline{c} & 1 \end{pmatrix}. \tag{7.2}$$

Since we assumed that $A$ was positive semidefinite, $\tilde{A}$ will be positive semidefinite if and only if its determinant is at least zero. This gives the necessary condition that

$$1 - |a|^2 - |c|^2 \geq 0. \tag{7.3}$$

By symmetry, this condition is necessary when $a$ or $c$ is replaced by 0 as well. So $A$ is one-fold invincible only if

$$\max\{|a|^2 + |b|^2, |a|^2 + |c|^2, |b|^2 + |c|^2\} \leq 1. \tag{7.4}$$

Since we have considered all possible first order gangster operators, this condition is also sufficient for one-fold invincibility.

Now, let us examine two-fold invincibility. $A$ will be two-fold invincible if and only if it is one-fold invincible and each of $1-|a|^2$, $1-|b|^2$, and $1-|c|^2$ are greater than zero. That this is the case can be seen by setting two of $a$, $b$, and $c$, equal to 0 in the condition (7.4), since a matrix is two-fold invincible if and only if it is one-fold invincible and any matrix obtained after applying a first order gangster operator to it is one-fold invincible. But the condition (7.4) implies that each of $1-|a|^2$, $1-|b|^2$, and $1-|c|^2$ are greater than or equal to 0. Since two-fold invincibility clearly implies three-fold invincibility in this case, we have the following result:

**Theorem 7.1.** *Let*

$$A = \begin{pmatrix} 1 & a & b \\ \overline{a} & 1 & c \\ \overline{b} & \overline{c} & 1 \end{pmatrix} \tag{7.5}$$

*be a positive semidefinite matrix. Then $A$ is one-fold invincible if and only if*

$$\max\{|a|^2+|b|^2, |a|^2+|c|^2, |b|^2+|c|^2\} \leq 1.$$

*Furthermore, one-fold invincibility implies full invincibility in this case.*

The second way to solve the general $3 \times 3$ case begins by noting that, by signature similarity, any real $3 \times 3$ matrix with positive diagonal entries can be made to have sign pattern

$$\begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \end{pmatrix} \text{ or } \begin{pmatrix} + & - & - \\ - & + & + \\ - & + & + \end{pmatrix}.$$

In the first case, the matrix is invincible if and only if it is PSD. The second case is easily taken care of using Theorem 4.2. For by symmetry we need only consider four plus-minus gangster operators: those with index sets $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. But these are the gangster operators

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. The last one always yields a PSD matrix. So in either of these two cases, assuming that the matrix is PSD, it suffices to check the gangster operators

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that a matrix signature similar to a given matrix remains PSD under some gangster operator if and only if the original matrix remains PSD under the same gangster operator. Thus, for any PSD matrix, we need only check those three gangster operators. This gives the same result for real matrices as the first method.

A third way to get the same result is to note that the graph of a non-reducible $3 \times 3$ matrix is either a tree, in which case we need only check that the matrix is PSD for invincibility, or a three cycle. But after removing any edge of a three cycle we get a tree, so we need only check the three $3 \times 3$ first order gangster operators.

In the $4 \times 4$ case, signature similarity gives three distinct sign patterns, *after permutation.* Only the last of these patterns is affected by permutation. The patterns are

$$\begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}, \begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}, \text{ and } \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}.$$

For the first pattern, by Corollary 4.3, we only have to check the gangster operators

$$\begin{gathered} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \end{gathered} \tag{7.6}$$

(We have omitted the identity matrix since, as a gangster operator, it always leaves any matrix with positive diagonal entries PSD.)

In the second case, we do not have to check any gangster operators.

For the third case, Theorem 4.2 shows that the following gangster operators (and only these gangster operators) must be checked:

$$\begin{gathered} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \end{gathered} \tag{7.7}$$

To get a list of all gangster operators one needs to check in the $4 \times 4$ case, one could take all gangster operators in (7.6) together with all permutations of all the gangster operators in (7.7). However, it is most likely easier to just convert the given matrices into one of the three sign patterns above using permutations and signature similarity, and then to apply the appropriate gangster operators.

## References

[1] Q. Zhao, S. E. Karisch, F. Rendl, H. Wolkowicz, Semidefinite programming relaxations for the quadratic assignment problem, J. Comb. Optim. 2 (1) (1998) 71–109, semidefinite programming and interior-point approaches for combinatorial optimization problems (Toronto, ON, 1996).

[2] H. Wolkowicz, Q. Zhao, Semidefinite programming relaxations for the graph partitioning problem, Discrete Appl. Math. 96/97 (1999) 461–479, the satisfiability problem (Certosa di Pontignano, 1996); Boolean functions.

[3] R. A. Horn, C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1990, corrected reprint of the 1985 original.

[4] R. A. Horn, C. R. Johnson, Topics in matrix analysis, Cambridge University Press, Cambridge, 1991.